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# Curvature tensors and Singer invariants of four-dimensional homogeneous spaces

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*Dedicated to the memory of Professor Yôsuke Ogawa*

*Abstract.* We show that the Singer invariant of a four-dimensional homogeneous space is at most 1.

*Keywords:* homogeneous spaces, Singer invariant

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## 1. Introduction

First of all we recall the theory of infinitesimally homogeneous spaces by I.M. Singer [14]. Let $(M, \langle,\rangle)$ be a Riemannian manifold of dimension $n$. We denote by $R$ and $\nabla^i R$, the curvature tensor and its $i$-th covariant derivative of $M$. Singer introduced the following condition:

$P(l)$: for every $p, q \in M$ there exists a linear isometry $\phi : T_pM \to T_qM$ such that

$$\phi^*(\nabla^i R)_q = (\nabla^i R)_p \quad i = 0, 1, \ldots, l.$$

A Riemannian manifold which satisfies $P(0)$ is said to be *curvature homogeneous* and if $P(l)$ holds, the manifold is said to be *curvature homogeneous up to order* $l$. We denote by $\mathfrak{so}(T_pM)$ the Lie algebra of the endomorphisms of $T_pM$ which are skew-symmetric with respect to $\langle,\rangle$. For a non-negative integer $l$, we define a Lie subalgebra $\mathfrak{g}_l(p)$ of $\mathfrak{so}(T_pM)$ by

$$\mathfrak{g}_l(p) = \{A \in \mathfrak{so}(T_pM) \mid A \cdot (\nabla^i R)_p = 0, \quad i = 0, 1, \ldots, l\},$$

where $A$ acts as a derivation on the tensor algebra on $T_pM$. Since $\mathfrak{g}_l(p) \supseteq \mathfrak{g}_{l+1}(p)$, there exists a first integer $k(p)$ such that $\mathfrak{g}_{k(p)}(p) = \mathfrak{g}_{k(p)+1}(p)$. Namely, we have

$$\mathfrak{so}(T_pM) \supseteq \mathfrak{g}_0(p) \supsetneq \mathfrak{g}_1(p) \supsetneq \mathfrak{g}_2(p) \supsetneq \cdots \supsetneq \mathfrak{g}_{k(p)}(p) = \mathfrak{g}_{k(p)+1}(p).$$

Following Singer, we say that $(M, \langle,\rangle)$ is *infinitesimally homogeneous* if $M$ satisfies $P(k(p)+1)$ for some point $p \in M$. If $M$ satisfies $P(l)$, then the linear isometry $\phi$ induces a Lie algebra isomorphism of $\mathfrak{g}_i(p)$ to $\mathfrak{g}_i(q)$ for $i = 0, 1, \ldots, l$. Therefore if $M$ is infinitesimally homogeneous, $k(q)$ does not depend on $q \in M$. We put

$k_M = k(p)$ and call it *the Singer invariant* of an infinitesimally homogeneous space $M$.

If $M$ is locally homogeneous, then evidently $M$ satisfies $P(l)$ for any $l$ and in particular $M$ is infinitesimally homogeneous. Singer proved the converse ([14] and see also L. Nicolodi and F. Tricerri [12]).

**Theorem.** *A connected infinitesimally homogeneous space $M$ is locally homogeneous. Moreover it is completely determined by its curvature tensor and covariant derivatives up to order $k_M + 1$.*

The theorem above suggests that the Singer invariant will play an important role in the differential geometry of locally homogeneous spaces. However, at our knowledge, there are only a few homogeneous spaces whose Singer invariants are known. So we investigate the Singer invariants of low dimensional homogeneous spaces.

For a three-dimensional homogeneous space the longest sequence which may occur is

$$\mathfrak{g}_0 = \mathfrak{so}(2) \supsetneq \mathfrak{g}_1 = \{0\}.$$

Therefore the Singer invariant of a three-dimensional homogeneous space is at most 1. F.G. Lastaria ([10], [11]) has computed $\mathfrak{g}_0$ and $\mathfrak{g}_1$ for three-dimensional homogeneous spaces and determined their Singer invariants (see also [6]).

In the case of dimension 4, it is known that every homogeneous space is either locally symmetric or locally isometric to a Lie group with a left invariant metric ([1], [4]). For the latter case it seems to be difficult to compute $\mathfrak{g}_i$ $(i = 0, 1, \dots)$ and to determine its Singer invariant by means of them. So, we will proceed in another way. Let $\mathcal{R}$ be the space of algebraic curvature tensors on $\mathbb{R}^4$. As is well known, $O(4)$ acts on $\mathcal{R}$ and $\dim \mathcal{R} = 20$. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{so}(4)$ which corresponds to a closed subgroup of $O(4)$. We denote by $\mathcal{R}^{\mathfrak{h}}$ the subspace of $\mathcal{R}$ which consists of curvature tensors invariant by $\mathfrak{h}$. We say that the curvature tensor of a four-dimensional homogeneous space $M$ belongs to $\mathcal{R}^{\mathfrak{h}}$ if there exists an orthonormal frame $u$ at $p \in M$ such that $u^* R_p \in \mathcal{R}^{\mathfrak{h}}$. Now our way is the following:

(1) Classify the $\mathfrak{h}$ which are isotropy subalgebras of the action of $\mathfrak{so}(4)$ on $\mathcal{R}$.

(2) With respect to each $\mathfrak{h}$ in the above, classify the homogeneous spaces whose curvature tensors belong to $\mathcal{R}^{\mathfrak{h}}$.

(3) Compute the Singer invariants of the homogeneous spaces obtained in (2).

In this paper we cannot completely do the approach above. But we show that if $\dim \mathfrak{h} \geq 2$, a homogeneous space whose curvature tensor belongs to $\mathcal{R}^{\mathfrak{h}}$ is locally symmetric or locally homothetic to $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2/SO(2)$ with the $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ invariant metric. Moreover, the Singer invariant of $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2/SO(2)$ is 1. By these results we obtain the following:

**Main result.** *The Singer invariant of a four-dimensional homogeneous space is at most* 1.

As for higher dimensional cases, it is worthwhile to remark the interesting examples constructed by O. Kowalski, F. Tricerri, and L. Vanhecke ([9]). They constructed a family of metrics on $\mathbb{R}^{n+1}$ which have the following properties:

(i) They are semi-symmetric (i.e., they satisfy $R(x,y)\cdot R=0$).

(ii) At every point $p$ of $\mathbb{R}^{n+1}$,

$$\begin{aligned}\mathfrak{g}_0(p)&=\mathfrak{so}(2)\oplus\mathfrak{so}(n-1),\\ \mathfrak{g}_l(p)&=\mathfrak{so}(n-1-l)\qquad \text{for } 1\le l\le n-3\ ,\\ \mathfrak{g}_{n-2}(p)&=0.\end{aligned}$$

Especially, their Singer invariants are equal to $n-2$.

(iii) They are not locally homogeneous.

Although they are not locally homogeneous, we think that they will give many suggestions when we consider the Singer invariants.

The content of the paper is organized as follows. In Section 2, we give a classification of $\mathfrak{h}$ (Proposition 2.1) and consider the cases of $\mathfrak{h}=\mathfrak{u}(2),\mathfrak{su}(2)$, and $\mathfrak{so}(3)$. In Section 3, we compute the curvature tensor and its covariant derivative of $SL(2,\mathbb{R})\ltimes\mathbb{R}^2/SO(2)$ with the $SL(2,\mathbb{R})\ltimes\mathbb{R}^2$ invariant metric and show that its Singer invariant is 1 (Corollary 3.2). In Section 4, we investigate the four-dimensional homogeneous spaces whose curvature tensors belong to $\mathcal{R}^{\mathfrak{t}}$, $\mathfrak{t}\cong\mathfrak{so}(2)\oplus\mathfrak{so}(2)$ and prove that they are locally symmetric or locally homothetic to $SL(2,\mathbb{R})\ltimes\mathbb{R}^2/SO(2)$ (Theorem 4.1). We remark that our main theorem is closely related to a result of K. Sekigawa, H. Suga, and L. Vanhecke ([13]) which has shown that a four-dimensional connected Riemannian manifold which is curvature homogeneous up to order one is locally homogeneous.

Finally we would like to thank Professor O. Kowalski and Professor L. Vanhecke for their valuable comments.

## 2. Classification of $\mathfrak{h}$

In order to state our proposition, we prepare some notations. Let $\{e_1,e_2,e_3,e_4\}$ be a standard basis of $\mathbb{R}^4$. We define a linear transformation $J$ of $\mathbb{R}^4$ by $Je_1=e_2$, $Je_2=-e_1$, $Je_3=e_4$, $Je_4=-e_3$. We define the Lie subalgebras $\mathfrak{u}(2)$ and $\mathfrak{su}(2)$ of $\mathfrak{so}(4)$ by

$$\begin{aligned}\mathfrak{u}(2)&=\{X\in\mathfrak{so}(4)\ \mid\ XJ=JX\}\,,\\ \mathfrak{su}(2)&=\{X\in\mathfrak{u}(2)\ \mid\ trace(XJ)=0\}\,.\end{aligned}$$

We denote by $\mathfrak{so}(3)$ the subalgebra of $\mathfrak{so}(4)$ consisting of all matrices of the forms:

$$\begin{pmatrix} & & & 0\\ & X & & 0\\ & & & 0\\ 0&0&0&0\end{pmatrix}\qquad (\,X:\ \text{skew-symmetric of degree } 3\,).$$

We define a maximal Abelian subalgebra $\mathfrak{t}$ and 1-dimensional subalgebras $\mathfrak{t}_{1,1}$, $\mathfrak{t}_{1,0}$ of $\mathfrak{so}(4)$ as follows:

$$\mathfrak{t} = \left\{ \begin{pmatrix} 0 & -\lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu \\ 0 & 0 & \mu & 0 \end{pmatrix} \mid \lambda, \mu \in \mathbb{R} \right\},$$

$$\mathfrak{t}_{1,1} = \{\lambda J \mid \lambda \in \mathbb{R}\},$$

$$\mathfrak{t}_{1,0} = \left\{ \begin{pmatrix} 0 & -\lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid \lambda \in \mathbb{R} \right\}.$$

Without difficulty we can obtain the following proposition.

**Proposition 2.1.** *An isotropy subalgebra of the action of $\mathfrak{so}(4)$ on $\mathcal{R}$ is conjugate under an adjoint transformation by an element of $O(4)$ to one of the following $\mathfrak{h}$:*

| $\mathfrak{h}$ | $\dim \mathcal{R}^{\mathfrak{h}}$ |
|---|---|
| $\mathfrak{so}(4)$ | 1 |
| $\mathfrak{u}(2)$ | 2 |
| $\mathfrak{su}(2)$ | 6 |
| $\mathfrak{so}(3)$ | 2 |
| $\mathfrak{t}$ | 4 |
| $\mathfrak{t}_{1,1}$ | 10 |
| $\mathfrak{t}_{1,0}$ | 6 |
| $\{0\}$ | 20 |

As the next step, we shall classify the four-dimensional homogeneous spaces whose curvature tensors belong to $\mathcal{R}^{\mathfrak{h}}$ for each $\mathfrak{h}$ in the proposition above. When $\mathfrak{h} = \mathfrak{so}(4)$, evidently they have constant sectional curvature. In this section, we consider the cases of $\mathfrak{h} = \mathfrak{u}(2), \mathfrak{su}(2)$ or $\mathfrak{so}(3)$. In Section 4, we shall treat with the case $\mathfrak{h} = \mathfrak{t}$. However we could not give answers for the remaining cases.

The cases of $\mathfrak{h} = \mathfrak{u}(2)$ or $\mathfrak{su}(2)$. In these cases the homogeneous spaces whose curvature tensors belong to $\mathcal{R}^{\mathfrak{h}}$ are Einstein spaces. G.R. Jensen ([5]) classified the four-dimensional homogeneous Einstein spaces. In particular, they are all locally symmetric.

The case of $\mathfrak{h} = \mathfrak{so}(3)$. We apply the following theorem of H. Takagi to this case.

**Theorem 2.2** (H. Takagi [15])**.** *Let $M$ be a curvature homogeneous conformally flat Riemannian manifold with $\dim M \geq 3$. Then $M$ is isometric to one of the following manifolds:*

(1) *A Riemannian manifold of constant curvature.*

(2) *A Riemannian manifold which is locally a product of a Riemannian manifold of constant curvature* $c$ $(\neq 0)$ *and a Riemannian manifold of constant curvature* $-c$.

(3) *A Riemannian manifold which is locally a product of a Riemannian manifold of constant curvature* $c$ $(\neq 0)$ *and a 1-dimensional one.*

*In particular,* $M$ *is locally symmetric.*

His original statement is slightly different. He assumed the homogeneity of $M$. However as it was pointed out in [2], [3] and [7], Takagi's proof used only the curvature homogeneity. Let $\mathcal{R}$ be the space of algebraic curvature tensors on $\mathbb{R}^n$ and $\mathfrak{so}(n-1)$ the subalgebra of $\mathfrak{so}(n)$ consisting of all matrices of the form:

$$\begin{pmatrix} & & & 0 \\ & X & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \qquad (\, X: \ \text{skew-symmetric of degree } n-1\,).$$

We denote by $\mathcal{R}^{\mathfrak{so}(n-1)}$ the subspace of $\mathcal{R}$ consisting of $\mathfrak{so}(n-1)$-invariant curvature tensors. We put $e_n = (0, \cdots, 0, 1)^t$ in $\mathbb{R}^n$. We denote by $V$ a subspace of $\mathbb{R}^n$ whose vectors are orthogonal to $e_n$. If $n \geq 4$, $R \in \mathcal{R}^{\mathfrak{so}(n-1)}$ has the following form: for $x, y, z \in V$

$$\begin{aligned} &R(x,y)z = \alpha\{\langle y,z\rangle x - \langle x,z\rangle y\},\\ &R(x,e_n)y = \beta\langle x,y\rangle e_n,\\ &R(x,e_n)e_n = -\beta x,\\ &\text{the others are zeros,} \end{aligned}$$

where $\alpha$, $\beta$ are some constants.
Therefore we see that the Weyl part of $R$ vanishes. By Theorem 2.2, we have the following.

**Corollary 2.3.** *Let* $M$ *be an* $n(\geq 4)$*-dimensional curvature homogeneous Riemannian manifold whose curvature tensor belongs to* $\mathcal{R}^{\mathfrak{so}(n-1)}$. *Then* $M$ *is a Riemannian manifold of constant curvature or locally isometric to a product of a Riemannian manifold of constant curvature* $c$ $(\neq 0)$ *and a 1-dimensional one. In particular,* $M$ *is locally symmetric.*

## 3. The curvature tensor of $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2/SO(2)$

Let $K = SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ be the semi-direct product of $SL(2,\mathbb{R})$ and $\mathbb{R}^2$ by the usual representation of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$. The element of $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ is expressed by $(a, x)$, $a \in SL(2,\mathbb{R})$, $x \in \mathbb{R}^2$. $H = SO(2)$ is naturally viewed as a subgroup of $SL(2,\mathbb{R})$ and hence of $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$. Namely, $H = \{(a,0) \mid a \in SO(2)\}$. In this section, we compute the curvature tensor and its covariant derivative of the homogeneous space $M = K/H = SL(2,\mathbb{R}) \ltimes \mathbb{R}^2/SO(2)$ with $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ invariant Riemannian metric and show that its Singer invariant is 1.

Let $\mathfrak{k} = \mathfrak{sl}(2,\mathbb{R}) + \mathbb{R}^2$ and $\mathfrak{h} = \mathfrak{so}(2)$ be the Lie algebra and its Lie subalgebra corresponding to $K$ and $H$ respectively. The element of $\mathfrak{k}$ is expressed by $(A, x)$, $A \in \mathfrak{sl}(2,\mathbb{R})$, $x \in \mathbb{R}^2$. Then we have

$$\mathfrak{h} = \left\{ \left( \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}, 0 \right) \ \middle| \ \lambda \in \mathbb{R} \right\}.$$

We define elements $e_1$, $e_2$, $e_3$, and $e_4$ in $\mathfrak{k}$ as follows:

$$e_1 = \left(0, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \qquad e_2 = \left(0, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right),$$
$$e_3 = \left(\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0\right), \quad e_4 = \left(\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0\right).$$

We denote by $\mathfrak{m}$ the subspace of $\mathfrak{k}$ spanned by $e_1$, $e_2$, $e_3$ and $e_4$. Then $\mathfrak{m}$ is $Ad(H)$ invariant in $\mathfrak{k}$. For $J_0 = \left( \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), 0 \right) \in \mathfrak{h}$, the matrix of $ad\ J_0$ with respect to the above basis $\{e_1, e_2, e_3, e_4\}$ in $\mathfrak{m}$ is given by

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}.$$

We define an inner product $\langle , \rangle$ on $\mathfrak{m}$ such that $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis. Because of the form of $ad\ J_0$ in $\mathfrak{m}$, the inner product $\langle , \rangle$ is invariant under $Ad(H)$. We induce the $K$-invariant Riemannian metric on $M$ corresponding to this inner product $\langle , \rangle$. For the Riemannian metric obtained as above, we use the same symbol $\langle , \rangle$. We compute the curvature tensor and its covariant derivative of $(M, \langle , \rangle)$. Let $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ be a dual basis of $\{e_1, e_2, e_3, e_4\}$ in $\mathfrak{m}$. $\theta_i \wedge \theta_j \cdot \theta_k \wedge \theta_l$ denotes the symmetric product of the 2-forms $\theta_i \wedge \theta_j$ and $\theta_k \wedge \theta_l$ defined by

$$\theta_i \wedge \theta_j \cdot \theta_k \wedge \theta_l = \frac{1}{2}\{\theta_i \wedge \theta_j \otimes \theta_k \wedge \theta_l + \theta_k \wedge \theta_l \otimes \theta_i \wedge \theta_j\}.$$

**Proposition 3.1.** *On the Riemannian manifold $(M, \langle , \rangle)$ the curvature tensor $R$, the Ricci tensor $\rho$ and the covariant derivative $\nabla R$ of the curvature tensor are expressed at the origin as follows:*

$$\begin{aligned} R = \frac{1}{4}\{ & -2\theta_1 \wedge \theta_2 \cdot \theta_1 \wedge \theta_2 + 4\theta_3 \wedge \theta_4 \cdot \theta_3 \wedge \theta_4 \\ & - 4\theta_1 \wedge \theta_2 \cdot \theta_3 \wedge \theta_4 - 2\theta_1 \wedge \theta_3 \cdot \theta_2 \wedge \theta_4 + 2\theta_1 \wedge \theta_4 \cdot \theta_2 \wedge \theta_3 \\ & + \theta_1 \wedge \theta_3 \cdot \theta_1 \wedge \theta_3 + \theta_1 \wedge \theta_4 \cdot \theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_3 \cdot \theta_2 \wedge \theta_3 + \theta_2 \wedge \theta_4 \cdot \theta_2 \wedge \theta_4\}, \end{aligned}$$

$$\rho = -\frac{3}{2}\{\theta_3 \otimes \theta_3 + \theta_4 \otimes \theta_4\},$$

$$\nabla R = \frac{3}{2}\{-\theta_1 \otimes \theta_1 \wedge \theta_2 \cdot (\theta_1 \wedge \theta_3 - \theta_2 \wedge \theta_4) + \theta_2 \otimes \theta_1 \wedge \theta_2 \cdot (\theta_2 \wedge \theta_3 + \theta_1 \wedge \theta_4)\}.$$

We denote by $\mathfrak{so}(\mathfrak{m})$ the Lie algebra of the endomorphisms of $\mathfrak{m}$ which are skew-symmetric with respect to $\langle , \rangle$. We express the elements of $\mathfrak{so}(\mathfrak{m})$ as the matrices with respect to the basis $\{e_1, e_2, e_3, e_4\}$ and identify $\mathfrak{so}(\mathfrak{m})$ with $\mathfrak{so}(4)$. From Proposition 3.1, we immediately obtain the following.

**Corollary 3.2.** *Let $\mathfrak{g}_0$ and $\mathfrak{g}_1$ be the Lie subalgebras of $\mathfrak{so}(\mathfrak{m})$ defined in § 1 for the homogeneous Riemannian manifold $(M, \langle , \rangle)$. Then we have $\mathfrak{g}_0 = \mathfrak{t}$ and $\mathfrak{g}_1 = ad\ \mathfrak{h}$, where $\mathfrak{t}$ is a maximal Abelian subalgebra of $\mathfrak{so}(4)$ defined in § 2. In particular, the Singer invariant of $(M, \langle , \rangle)$ is* 1.

**Remark 3.3.** It is easy to show that a Riemannian manifold $M = K/H = SL(2,\mathbb{R}) \ltimes \mathbb{R}^2/SO(2)$ with a $K$-invariant Riemannian metric is homothetic to $(M, \langle , \rangle)$ with the Riemannian metric defined as above. Therefore the Singer invariant of the homogeneous space $K/H$ with any $K$-invariant metric is 1.

**Remark 3.4.** We put $E_1, E_2, E_3, E_4$ in the Lie algebra $\mathfrak{k} = \mathfrak{sl}(2,\mathbb{R}) + \mathbb{R}^2$ as $E_i = e_i$ $(i = 1, 2, 4)$, $E_3 = e_3 - \frac{1}{2}J_0 = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0\right)$. Then we have

$$\begin{aligned} &[E_1, E_2] = 0, && [E_1, E_3] = 0, && [E_1, E_4] = -\frac{1}{2}E_1 \\ &[E_2, E_3] = -E_1, && [E_2, E_4] = \frac{1}{2}E_2, && [E_3, E_4] = -E_3. \end{aligned}$$

Therefore the subspace $\mathfrak{g}$ spanned by $E_1$, $E_2$, $E_3$, and $E_4$ is a Lie subalgebra of $\mathfrak{k}$. We denote by $G$ the connected Lie subgroup of $K$ corresponding to $\mathfrak{g}$. Then $G$ acts simply transitively on $M$. Moreover, the Riemannian manifold $M$ is isometric to the Lie group $G$ with the left invariant metric such that $E_1$, $E_2$, $E_3$, and $E_4$ are orthonormal.

## 4. Four-dimensional homogeneous spaces whose curvature tensors belong to $\mathcal{R}^{\mathfrak{t}}$

Let $\mathfrak{t}$ be the maximal Abelian subalgebra of $\mathfrak{so}(4)$ given in § 2 and $\mathcal{R}^{\mathfrak{t}}$ be the subspace of $\mathcal{R}$ consisting of algebraic curvature tensors invariant by $\mathfrak{t}$.

**Theorem 4.1.** *Let $M$ be a four-dimensional homogeneous space whose curvature tensor belongs to $\mathcal{R}^{\mathfrak{t}}$. Then $M$ is either locally symmetric or locally homothetic to $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2/SO(2)$ with the $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ invariant Riemannian metric given in § 3.*

The purpose of this section is to prove the theorem above. We take curvature tensors $R_1$, $R_2$, $R_3$, $R_4$ of the following form:

$R_1 = \theta_1 \wedge \theta_2 \cdot \theta_1 \wedge \theta_2,$

$R_2 = \theta_3 \wedge \theta_4 \cdot \theta_3 \wedge \theta_4,$

$R_3 = 2\theta_1 \wedge \theta_2 \cdot \theta_3 \wedge \theta_4 + \theta_1 \wedge \theta_3 \cdot \theta_2 \wedge \theta_4 - \theta_1 \wedge \theta_4 \cdot \theta_2 \wedge \theta_3,$

$R_4 = \theta_1 \wedge \theta_3 \cdot \theta_1 \wedge \theta_3 + \theta_1 \wedge \theta_4 \cdot \theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_3 \cdot \theta_2 \wedge \theta_3 + \theta_2 \wedge \theta_4 \cdot \theta_2 \wedge \theta_4,$

where $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ is a dual basis of the standard basis in $\mathbb{R}^4$. Then $\{R_1, R_2, R_3, R_4\}$ is a basis of $\mathcal{R}^{\mathfrak{t}}$.

**Remark 4.2.** Let $S_i$ $(i = 1, 2)$ be elements of $O(4)$ given by

$$S_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then $S_i(\mathcal{R}^{\mathfrak{t}}) = \mathcal{R}^{\mathfrak{t}}$. Moreover, we have

$$S_1(\alpha R_1 + \beta R_2 + \gamma R_3 + \delta R_4) = \beta R_1 + \alpha R_2 + \gamma R_3 + \delta R_4,$$
$$S_2(\alpha R_1 + \beta R_2 + \gamma R_3 + \delta R_4) = \alpha R_1 + \beta R_2 - \gamma R_3 + \delta R_4.$$

For the subalgebra $\mathfrak{u}(2)$ of $\mathfrak{so}(4)$ given in § 2, we put $\mathfrak{u}(2)' = Ad(S_2)\mathfrak{u}(2)$. Lie subalgebras of $\mathfrak{so}(4)$ which contain $\mathfrak{t}$ are $\mathfrak{u}(2)$, $\mathfrak{u}(2)'$ and $\mathfrak{so}(4)$. The following is easily seen.

**Lemma 4.3.** *Let $R = \alpha R_1 + \beta R_2 + \gamma R_3 + \delta R_4$ be a curvature tensor of $\mathcal{R}^{\mathfrak{t}}$. Then we have*

(i) $R \in \mathcal{R}^{\mathfrak{u}(2)}$ *if and only if* $\alpha = \beta = \frac{3}{2}\gamma + \delta$,
(ii) $R \in \mathcal{R}^{\mathfrak{u}(2)'}$ *if and only if* $\alpha = \beta = -\frac{3}{2}\gamma + \delta$,
(iii) $R \in \mathcal{R}^{\mathfrak{so}(4)}$ *if and only if* $\gamma = 0$, $\alpha = \beta = \delta$.

Let $M$ be a four-dimensional homogeneous space. It is known that $M$ is locally symmetric or locally isometric to a Lie group $G$ with a left invariant Riemannian metric $\langle , \rangle$. To prove Theorem 4.1, it is sufficient to consider the latter case. Let $e_i$ $(i = 1, 2, 3, 4)$ be left invariant vector fields on a Lie group $G$ such that $\{e_1, e_2, e_3, e_4\}$ is an orthonormal frame field. We assume that with respect to this frame, the curvature tensor of $(G, \langle , \rangle)$ belongs to $\mathcal{R}^{\mathfrak{t}}$ and it has the form $R = \alpha R_1 + \beta R_2 + \gamma R_3 + \delta R_4$. The Riemannian connection $\nabla$ is described by

$$\nabla_{e_i} e_j = \sum_{k=1}^{4} {\Gamma_{ij}}^k e_k \quad (i, j = 1, 2, 3, 4),$$

where ${\Gamma_{ij}}^k$ are constants. For each $i$, $\Gamma_i = \left({\Gamma_{ij}}^k\right)$ is a skew-symmetric matrix of degree 4, that is, $\Gamma_i \in \mathfrak{so}(4)$. The covariant derivative $\nabla R$ of the curvature tensor is given by $\nabla_{e_i} R = \Gamma_i \cdot R$, where $\Gamma_i$ acts as a derivation. From the second Bianchi identity, it follows that

$$\mathfrak{S}_{i,j,k} (\Gamma_i \cdot R) (e_j, e_k, e_l, e_u) = 0,$$

where $\mathfrak{S}$ denotes the cyclic sum. This gives a system of linear equations for unknown numbers ${\Gamma_{ij}}^k$ (cf. O. Kowalski and F. Prüfer [8]). From these, we immediately obtain the following.

**Lemma 4.4.** *If* $\left(\frac{3}{2}\gamma\right)^2 - (\alpha-\delta)^2 \neq 0$, *then we have* $\Gamma_{i1}^{\ 3} = \Gamma_{i1}^{\ 4} = \Gamma_{i2}^{\ 3} = \Gamma_{i2}^{\ 4} = 0$ *for* $i = 3, 4$. *If* $\left(\frac{3}{2}\gamma\right)^2 - (\beta-\delta)^2 \neq 0$, *then we have* $\Gamma_{i1}^{\ 3} = \Gamma_{i1}^{\ 4} = \Gamma_{i2}^{\ 3} = \Gamma_{i2}^{\ 4} = 0$ *for* $i = 1, 2$.

By Lemma 4.4, we shall consider the following cases:

(I) $\left(\frac{3}{2}\gamma\right)^2 - (\alpha-\delta)^2 \neq 0$, $\left(\frac{3}{2}\gamma\right)^2 - (\beta-\delta)^2 \neq 0$,

(II)-(i) $\left(\frac{3}{2}\gamma\right)^2 - (\alpha-\delta)^2 \neq 0$, $\beta - \delta = \frac{3}{2}\gamma \neq 0$,

(II)-(ii) $\gamma = 0$, $\alpha \neq \beta = \delta$,

(III)-(i) $\alpha - \delta = \beta - \delta = \frac{3}{2}\gamma \neq 0$,

(III)-(ii) $\alpha - \delta = -(\beta - \delta) = \frac{3}{2}\gamma \neq 0$,

(III)-(iii) $\gamma = 0$, $\alpha = \beta = \delta$.

We note that it is sufficient to consider only the cases above owing to Remark 4.2. In the cases of (III)-(i) and (III)-(iii) we have $R \in \mathcal{R}^{\mathfrak{u}(2)}$ and $R \in \mathcal{R}^{\mathfrak{so}(4)}$, respectively by Lemma 4.3 and they have already been discussed in § 2. Therefore it suffices to consider the cases (I), (II)-(i), (ii), and (III)-(ii).

**Lemma 4.5.** *In the case* (I), *we have* $\gamma = \delta = 0$ *and* $G$ *is locally isometric to the Riemannian product of two surfaces of constant negative curvatures* $-\alpha$ *and* $-\beta$.

Proof of Lemma 4.5: By Lemma 4.4, we have $\Gamma_{i1}^{\ 3} = \Gamma_{i1}^{\ 4} = \Gamma_{i2}^{\ 3} = \Gamma_{i2}^{\ 4} = 0$ for $i = 1, 2, 3, 4$. These imply that $\nabla R = 0$. Let $\mathcal{D}_1$ (resp. $\mathcal{D}_2$) be the distribution generated by the vector fields $e_1$ and $e_2$ (resp. $e_3$ and $e_4$). The identities above mean that $\mathcal{D}_1$ and $\mathcal{D}_2$ are parallel distributions. In particular we have $\gamma = 2\langle R(e_1, e_3)e_2, e_4\rangle = 0$ and $\delta = \langle R(e_1, e_3)e_1, e_3\rangle = 0$. By the decomposition theorem of de Rham, we see that $G$ is locally isometric to the Riemannian product of two surfaces of constant curvatures $-\alpha$ and $-\beta$. By further computations, it follows that $\alpha$ and $\beta$ are positive. □

**Remark 4.6.** Lemma 4.5 holds under the condition of the curvature homogeneity. Accordingly there does not exist a curvature homogeneous Riemannian manifold whose curvature tensor belongs to $\mathcal{R}^{\mathfrak{t}}$ and satisfies $\left(\frac{3}{2}\gamma\right)^2 - (\alpha-\delta)^2 \neq 0$, $\left(\frac{3}{2}\gamma\right)^2 - (\beta-\delta)^2 \neq 0$ and $\gamma \neq 0$ or $\delta \neq 0$. The curvature tensors which satisfy these conditions are generic in $\mathcal{R}^{\mathfrak{t}}$.

**Lemma 4.7.** *The case* (II)-(i) *does not occur.*

**Lemma 4.8.** *In the case* (II)-(ii), *we have* $\beta = \delta = 0$ *and* $G$ *is locally isometric to the Riemannian product of a flat surface and a surface of constant negative curvature* $-\alpha$.

We omit the proofs of Lemmas 4.7 and 4.8, because they are similar to that of the next lemma.

**Lemma 4.9.** *In the case* (III)-(ii), *$G$ is locally homothetic to $SL(2,\mathbb{R})\ltimes\mathbb{R}^2/SO(2)$ with the $SL(2,\mathbb{R})\ltimes\mathbb{R}^2$ invariant Riemannian metric given in § 3.*

Proof of Lemma 4.9: The second Bianchi identity gives a system of linear equations for unknown numbers ${\Gamma_{ij}}^k$. Moreover, computing each component of the curvature tensor, we obtain a system of quadratic equations for ${\Gamma_{ij}}^k$. Solving these equations, we have the following two cases:

Case 1
$$-{\Gamma_{11}}^3={\Gamma_{12}}^4={\Gamma_{21}}^4={\Gamma_{22}}^3 \quad \text{(which we denote by } A\text{)},$$
$${\Gamma_{11}}^4={\Gamma_{12}}^3={\Gamma_{21}}^3=-{\Gamma_{22}}^4 \quad \text{(which we denote by } B\text{)},$$
$${\Gamma_{i1}}^3={\Gamma_{i1}}^4={\Gamma_{i2}}^3={\Gamma_{i2}}^4=0 \quad (i=3,4),$$
$${\Gamma_{11}}^2={\Gamma_{21}}^2={\Gamma_{13}}^4={\Gamma_{23}}^4=0,$$
$$2{\Gamma_{31}}^2+{\Gamma_{33}}^4=0, \quad 2{\Gamma_{41}}^2+{\Gamma_{43}}^4=0,$$
$$-\frac{1}{2}\alpha=\frac{1}{4}\beta=-\frac{1}{2}\gamma=\delta=\left({\Gamma_{31}}^2\right)^2+\left({\Gamma_{41}}^2\right)^2=A^2+B^2>0,$$

Case 2
$${\Gamma_{31}}^3={\Gamma_{32}}^4=-{\Gamma_{41}}^4={\Gamma_{42}}^3 \quad \text{(which we denote by } C\text{)},$$
$${\Gamma_{31}}^4=-{\Gamma_{32}}^3={\Gamma_{41}}^3={\Gamma_{42}}^4 \quad \text{(which we denote by } D\text{)},$$
$${\Gamma_{i1}}^3={\Gamma_{i1}}^4={\Gamma_{i2}}^3={\Gamma_{i2}}^4=0 \quad (i=1,2),$$
$${\Gamma_{31}}^2={\Gamma_{41}}^2={\Gamma_{33}}^4={\Gamma_{43}}^4=0,$$
$$2{\Gamma_{13}}^4-{\Gamma_{11}}^2=0, \quad 2{\Gamma_{23}}^4-{\Gamma_{21}}^2=0,$$
$$\frac{1}{4}\alpha=-\frac{1}{2}\beta=\frac{1}{2}\gamma=\delta=\left({\Gamma_{13}}^4\right)^2+\left({\Gamma_{23}}^4\right)^2=C^2+D^2>0.$$

For Case 1, the bracket operations $[,]$ have the form:

$$\begin{aligned}
[e_1,e_2]&=0,\\
[e_1,e_3]&=Ae_1-\left(B+{\Gamma_{31}}^2\right)e_2,\\
[e_1,e_4]&=-Be_1-\left(A+{\Gamma_{41}}^2\right)e_2,\\
[e_2,e_3]&=\left(-B+{\Gamma_{31}}^2\right)e_1-Ae_2,\\
[e_2,e_4]&=\left(-A+{\Gamma_{41}}^2\right)e_1+Be_2,\\
[e_3,e_4]&={\Gamma_{34}}^3e_3-{\Gamma_{43}}^4e_4.
\end{aligned}$$

By a suitable change of an orthonormal frame field $\{e_1,e_2,e_3,e_4\}$, we have

$$\begin{aligned}
&[e_1,e_2]=0, && [e_1,e_3]=0, && [e_1,e_4]=-te_1,\\
&[e_2,e_3]=-2te_1, && [e_2,e_4]=te_2, && [e_3,e_4]=-2te_3,
\end{aligned}$$

where $t = \sqrt{\delta} > 0$.

Noticing Remark 3.4, we see that $G$ is locally homothetic to $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2/SO(2)$ with the $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ invariant Riemannian metric. In Case 2, the same result holds. $\square$

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