$C_p(I)$ is not subsequential

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Abstract. If a separable dense in itself metric space is not a union of countably many nowhere dense subsets, then its C_p-space is not subsequential.

Keywords: C_p-space, sequential, subsequential

Classification: Primary 54A35, 03E35; Secondary 54A25

0. Introduction

A subspace of a sequential space is called subsequential. Some time ago A.V. Arhangel'skii asked if $C_p(I)$ is subsequential. In [2] the author gave an example of a countable space which is not subsequential but can be embedded as a subspace in $C_p(2^\omega)$. In this note we prove several general propositions concerning non subsequentiality of C_p-spaces. We also give two simple examples of nonsubsequential subspaces of $C_p(2^\omega)$.

Recall that $C_p(X)$ denotes the space of real-valued continuous functions on X with pointwise convergence topology, I denotes the usual segment $[0,1]$. It is well known that $C_p(I)$ is not sequential (see, for example [1]).

The following proposition is in fact due to E.G. Pytke'ev [3].

Proposition 0.1. *Let X be subsequential, $x \notin A$, $x \in \overline{A}$. Then there exists a countable π-network at x of infinite subsets of A, i.e. there exists at x a countable family $\mathcal{A}$ of infinite subsets of A such that each neighbourhood of x contains an element of $\mathcal{A}$.*

1. Propositions

Here we prove that very often a C_p-space is not subsequential.

Proposition 1.1. *Let X be a separable metric space and $\mathcal{P}$ a countable family of infinite subsets of X. Then there exists an open ω-cover $\mathcal{V}$ of X with the property*

($\mathcal{P}_s$). Suppose $\mathcal{K}$ is an infinite subfamily of $\mathcal{V}$, then $\bigcap\{\overline{V} : V \in \mathcal{K}\}$ does not contain any element of $\mathcal{P}$.

Proof: We can assume that the metric d of X is totally bounded, i.e. for every $\delta > 0$ there exists a finite cover of balls of diameter less than δ. Let $\{P_i : i \in \omega\}$ be an enumeration of elements of $\mathcal{P}$. Now we need a very simple

Lemma 1.2. *Suppose M is an infinite subset of a metric space. Then for every $n \in \omega$ there exists $\delta > 0$ such that M cannot be covered by a union of n balls of diameter less than δ.*

Proof of Lemma: Let d be the metric on the space under consideration. As M is infinite, we can find an $N \subset M$, $|N| = n+1$. Then $\delta = \min\{d(x,y) : x, y \in N, x \neq y\}$ is the desired number. □

Further, using this lemma we can construct a decreasing sequence of positive reals δ_i, $i \in \omega$, such that for every $i \in \omega$ every P_k, $k \leq i$, cannot be covered by a union of i closed balls of diameter less than δ_i. Now we find a sequence of finite open covers $\mathcal{W}_i$, $i \in \omega$, of balls of diameter less than δ_i. Further let $\mathcal{V}_i = \{\bigcup T : T \subset \mathcal{W}_i, |T| \leq i\}$. It is clear that $\mathcal{V}_i$ is finite. Let $\mathcal{V} = \bigcup\{\mathcal{V}_i : i \in \omega\}$. Let us prove that $\mathcal{V}$ is an open ω-cover of X with property $(\mathcal{P}_s)$. Let Z be a finite subset of X. Let us take an $i \in \omega$, $i \geq |Z|$. There is an element of $\mathcal{V}_i$ that covers Z. We proved that $\mathcal{V}$ is an ω-cover of X. Now let us finish the proof of Proposition 1.1. Let $P_k \in \mathcal{P}$. If $T \in \mathcal{V}$ and $T \supset P_k$, then $T \in \mathcal{V}_i$ with $i \leq k$. So, there are only finitely many elements of $\mathcal{V}$ that contain the given P_k. The proof of 1.1 is complete. □

Proposition 1.3. *Let X be a separable metric space. Let $\mathcal{P}$ be a countable family of infinite subsets of X. Then $C_p(X)$ has an infinite subspace F, $1 \notin F$, $1 \in \overline{F}$ with the property*

$(\mathcal{P}_c)$. Suppose K is an infinite subset of F, then $\bigcap\{f^{-1}[1/2, 3/2] : f \in K\}$ does not contain any element of $\mathcal{P}$.

Proof: Let $\{P_i : i \in \omega\}$ be an enumeration of elements of $\mathcal{P}$ and let $\{V_i : i \in \omega\}$ be an enumeration of elements of $\mathcal{V}$ from Proposition 1.1. It is clear from the proof of Proposition 1.1 that there is a function $f : \omega \to \omega$ such that $P_k \not\subset \overline{V}_i$ if $i \geq f(k)$. For every $i \in \omega$ we can easily construct a real-valued continuous function f_i such that $f_i^{-1}(1) \supset V_i$ and $P_k \not\subset f_i^{-1}[1/2, 3/2]$ for every $i \geq f(k)$. □

Now it remains to check that $F = \{f_i : i \in \omega\}$ is the desired subset of $C_p(X)$.

Proposition 1.4. *Let X be a space which is not a union of countably many nowhere dense subsets, let X have a countable π-network $\mathcal{N}$ of infinite subsets. If $C_p(X)$ has a subspace F from Proposition 1.3 with the property $(\mathcal{N}_c)$, then $C_p(X)$ is not subsequential.*

Proof: We have $1 \in \overline{F}$. Let us prove that 1 has no countable π-network of infinite subsets of F. Let us suppose the contrary and let $\{P_j : j \in \omega\}$ be such a π-net. Let $O_x[1, \epsilon)$ denote a basic neighbourhood of 1 in $C_p(X)$, i.e. $O_x[1, \epsilon) = \{f \in C_p(X) : |f(x) - 1| < \epsilon\}$. Then for every $x \in X$ there is a $j_x \in \omega$ such that $P_{j_x} \subset O_x[1/2, 3/2]$. As X is not a union of countably many nowhere dense subsets, there exist $m \in \omega$ and $X_m \subset X$ such that X_m is not nowhere dense and $m = j_x$ for each $x \in X_m$. It is clear that $\overline{X}_m \subset f^{-1}[1/2, 3/2]$ for every $f \in P_m$, hence $\overline{X}_m \subset \bigcap\{f^{-1}[1/2, 3/2] : f \in P_m\}$. But $Int(\overline{X}_m) \neq \emptyset$, hence

$\overline{X}_m$ contains some element $N' \in \mathcal{N}$. Then $N' \subset (\bigcap\{f^{-1}[1/2, 3/2] : f \in P_m\})$. A contradiction is obtained. $\square$

Combining Propositions 1.1, 1.3, 1.4 we obtain

Theorem 1.5. *If a separable dense in itself metric space is not a union of countably many nowhere dense subsets then its C_p-space is not subsequential.*

PROOF: It is enough to mention that a separable dense in itself metric space has a countable π-network of infinite subsets (moreover it has a countable base of nonempty open subsets which are infinite). $\square$

Corollary 1.6. *$C_p(I)$ is not subsequential, $C_p(Y)$ is not subsequential for a nonscattered compactum Y.*

PROOF: A compactum Y is not scattered iff it maps continuously onto I. But in this case $C_p(I) \subset C_p(Y)$. $\square$

Proposition 1.7 (Compact dixotomy)**.** *A C_p-space over a compactum either is sequential or is not subsequential.*

Corollary 1.8. *If a metric space contains a copy of 2^ω then its C_p-space is not subsequential. In particular, the C_p-space over an uncountable A-set in a metric space is not subsequential.*

Because in these cases C_p-space contains a copy of $C_p(2^\omega)$.

Theorem 1.5 and Corollary 1.8 allow us to raise the following conjecture:

Hypothesis 1.9 (General dixotomy)**.** *A C_p-space either is sequential or is not subsequential.*

2. Two concrete examples

Here we give two examples of nonsubsequential subspaces of $C_p(2^\omega)$.

2.1. The first example. It is the space Z introduced in [4]. We describe it here. Let $\{K_n : n \in \omega\}$ be disjoint finite subsets, $K = \bigcup\{K_n : n \in \omega\}$ and $* \notin K$. Let $Z = \{*\} \cup K$. Let all points of K be isolated and a typical neihgbourhood of $*$ be a set $\{*\} \cup (K \setminus L)$ where $|L \cap K_n| \leq m$ with the same m for every $k \in \omega$. In [2] it is proved that $*$ has no countable π-net of infinite subsets of K and it is proved that Z can be embedded as a subspace in $C_p(2^\omega)$.

2.2. The second example. We will work in 2^ω. Let us follow the general way described in Propositions 1.1, 1.3, 1.4. Let Ω_n denote the set of functions $f : n \to 2$ and let $\Omega = \bigcup\{\Omega_n : n \in \omega\}$. For every $f \in \Omega$ the subset $O(f) = \{x \in 2^\omega : x \supset f\}$ is a basic clopen subset in 2^ω.

For every $n \in \omega$, let $\mathcal{S}_n$ be the family $\{O(f) : f \in \Omega_{2^n}\}$ and $\mathcal{V}_n = \{\bigcup T : T \subset \mathcal{S}_n, |T| = n\}$.

Further, let $\mathcal{V} = \bigcup\{\mathcal{V}_n : n \in \omega\}$. A little later we will prove that $\mathcal{V}$ is a clopen ω-cover of 2^ω with the following property:

Suppose $\mathcal{K}$ is a infinite subfamily of $\mathcal{V}$, then $Int(\bigcap\mathcal{K}) = \emptyset$.

It implies that a subspace F of characteristic functions of elements of this cover $\mathcal{V}$ is the same as in Proposition 1.3. Hence this subspace demonstrates nonsubsequentiality of $C_p(2^\omega)$.

Now the desired proof. Let Z be a finite subset of 2^ω. Let us take some $n \geq |Z|$. As $\mathcal{S}_n$ covers 2^ω, there is an element of $\mathcal{V}_n$ that contains Z. Now let W be a clopen subset of 2^ω. For our goal we can assume that $W = O(f)$ for some $f \in \Omega_n$. We see that $m(O(f)) = 2^{-n}$ and $m(W) = i * 2^{-2^i}$ for a $W \in \mathcal{V}_i$. Here m denotes Lebesgue measure on 2^ω. Therefore if $W \supset O(f)$ then $i \leq n$, i.e. only finitely many elements of $\mathcal{V}$ contain (f).

Acknowledgment. The author would like to express the gratitude to M. Matveev for the help in the preparation of this article and to the referee for many helpful remarks.

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(*Received* December 1, 1997, *revised* January 8, 1999)