# A remark on associative copulas

Piotr Mikusiński, M.D. Taylor

*Abstract.* A method for producing associative copulas from a binary operation and a convex function on an interval is described.

*Keywords:* copulas, associative copulas, Archimedean copulas
*Classification:* 60E05, 62E10

Let $I$ denote the unit interval $[0,1]$. Copulas are cumulative distribution functions on $I^2$ with uniform marginals; more precisely, a *copula* is a function $C(x,y)$ on $I^2$ that satisfies

(1) (Boundary Conditions)

$$C(x,0) = C(0,y) = 0,\ C(x,1) = x \ \text{ and } \ C(1,y) = y \ \text{ for all } \ x,y \in I,$$

and

(2) (Monotonicity)

$$C(x_2,y_2) - C(x_1,y_2) - C(x_2,y_1) + C(x_1,y_1) \geq 0,$$

if $0 \leq x_1 \leq x_2 \leq 1$ and $0 \leq y_1 \leq y_2 \leq 1$.

For $\varphi$, a continuous, strictly decreasing function from $I$ to $[0,\infty]$ such that $\varphi(1) = 0$, we define the *pseudo-inverse* of $\varphi$ to be the function $\varphi^{[-1]} : [0,\infty] \to I$ defined by

$$\varphi^{[-1]}(x) = \begin{cases} \varphi^{-1}(x), & \text{if } \ 0 \leq x \leq \varphi(0), \\ 0, & \text{if } \ \varphi(0) \leq x \leq \infty. \end{cases} \tag{1}$$

We say that $C$ is an *Archimedean* copula with additive generator $\varphi$ provided that it is a copula and that there exists a function $\varphi$ of the type described here such that

$$C(x,y) = \varphi^{[-1]}(\varphi(x) + \varphi(y)). \tag{2}$$

To quote from [2], "These copulas find a wide range of applications for a number of reasons: (1) The ease with which they can be constructed; (2) The great variety of families of copulas which belong to this class; and (3) The many nice properties possessed by the members of this class." One of the most salient of these properties is that $C$ is associative, that is,

$$C(x, C(y,z)) = C(C(x,y), z).$$

We have the following characterization of Archimedean copulas:

**Theorem 1.** *Let $\varphi$ be a continuous, strictly decreasing function from $I$ to $[0,\infty]$ such that $\varphi(1)=0$. Then the function $C$ defined by (2) is a copula if and only if $\varphi$ is convex.*

Proof of this theorem can be found in [1] and [2]. Discussion of related Archimedean binary operations can be found in [3].

We show this theorem can be generalized in a simple and elegant fashion in which, instead of dealing with pseudo-inverses, we extend the notion of convexity.

Let $\oplus$ be a continuous associative operation in $[0,a]$, $a\in[0,\infty]$, such that $t\oplus 0 = 0\oplus t = t$ and $t\oplus a = a\oplus t = a$ for all $t\in[0,a]$.

**Example 2.** Let $a=\infty$ and $\oplus$ by the ordinary addition extended to $[0,\infty]$ in the obvious way. Clearly, the above conditions are satisfied.

**Example 3.** Let $a\in[0,\infty]$ be arbitrary and let $\oplus$ be defined by $s\oplus t=\max(s,t)$. Again, it is easy to check that the above conditions are satisfied.

**Example 4.** Let $a\in[0,\infty]$ be arbitrary and let $\oplus$ be defined by $s\oplus t=\min(s+t,a)$. Simple argument shows that the above conditions are satisfied.

A function $\psi:[0,a]\to\mathbb{R}$ is called $\oplus$-*convex* if

$$\psi(r\oplus t)-\psi(r)\le\psi(s\oplus t)-\psi(s) \tag{3}$$

for every $r\le s$ and any $t$.

**Lemma 5.** *If $\oplus$ is ordinary addition and $\psi$ is continuous, then $\psi$ satisfies* (3) *if and only if $\psi$ is convex.*

**Lemma 6.** *If $s\oplus t=\max(s,t)$, then $\psi$ satisfies is $\oplus$-convex if and only if $\psi$ is decreasing.*

Proof: In order to show that (3) implies that $\psi$ is decreasing it suffices to take $t=s$.

Now consider $r\le s$. We consider three cases. If $t\le r$, then (3) becomes

$$\psi(r)-\psi(r)\le\psi(s)-\psi(s),$$

which is always true. If $r\le t\le s$, then (3) reduces to

$$\psi(t)-\psi(r)\le\psi(s)-\psi(s)=0,$$

which is true since $\psi$ is decreasing. Finally, if $s\le t$, then (3) becomes

$$\psi(t)-\psi(r)\le\psi(t)-\psi(s),$$

or

$$\psi(s)\le\psi(r),$$

which is again true since $\psi$ is decreasing. □

Now we prove the main theorem of this note.

**Theorem 7.** *Let $\oplus$ be a continuous associative operation in $[0,a]$, $a \in [0,\infty]$, such that $t \oplus 0 = 0 \oplus t = t$ and $t \oplus a = a \oplus t = a$ for all $t \in [0,a]$. Let $\varphi : [0,1] \to [0,a]$ be a strictly decreasing continuous surjection. Define*

$$C(x,y) = \varphi^{-1}\left(\varphi(x) \oplus \varphi(y)\right). \tag{4}$$

*Then*
(a) *$C(0,z) = C(z,0) = 0$ and $C(1,z) = C(z,1) = z$ for all $z \in [0,1]$,*
(b) *$C$ is associative,*
(c) *$C$ is a copula if and only if $\varphi^{-1}$ is $\oplus$-convex.*

Parts (a) and (b) are easy. Before we prove part (c) we prove the following lemma.

**Lemma 8.** *Let $\oplus$, $\varphi$, and $C$ be as in Theorem 7. Then $C$ is monotonic if and only if*

$$C(u_2,v) - C(u_1,v) \le u_2 - u_1 \quad \text{whenever} \quad u_1 \le u_2. \tag{5}$$

Proof: Since every copula satisfies (5), it suffices to prove that (5) implies that $C$ is monotonic.

Assume (5) and consider $v_1 \le v_2$. Since $\varphi$ and $\oplus$ are continuous and $\varphi(v_2) \le \varphi(v_1)$, there exists $t \in [0,1]$ such that

$$\varphi(v_2) \oplus \varphi(t) = \varphi(v_1).$$

Hence

$$\begin{aligned}
&C(u_2,v_1) - C(u_1,v_1)\\
&\qquad = \varphi^{-1}\left(\varphi(u_2) \oplus \varphi(v_1)\right) - \varphi^{-1}\left(\varphi(u_1) \oplus \varphi(v_1)\right)\\
&\qquad = \varphi^{-1}\left(\varphi(u_2) \oplus (\varphi(v_2) \oplus \varphi(t))\right) - \varphi^{-1}\left(\varphi(u_1) \oplus (\varphi(v_2) \oplus \varphi(t))\right)\\
&\qquad = \varphi^{-1}\left((\varphi(u_2) \oplus \varphi(v_2)) \oplus \varphi(t)\right) - \varphi^{-1}\left((\varphi(u_1) \oplus \varphi(v_2)) \oplus \varphi(t)\right)\\
&\qquad = C(C(u_2,v_2),t) - C(C(u_1,v_2),t)\\
&\qquad \le C(u_2,v_2) - C(u_1,v_2),
\end{aligned}$$

which proves that $C$ is monotonic. □

Proof of part (c) in Theorem 7: Suppose $C$ is monotonic. Let $r \le s$ and $t > 0$. Let $u_1 = \varphi^{-1}(s)$, $u_2 = \varphi^{-1}(r)$, and $v = \varphi^{-1}(t)$. Since $u_1 \le u_2$, by Lemma 8, we have (5) and consequently

$$\varphi^{-1}(r \oplus t) - \varphi^{-1}(s \oplus t) \le \varphi^{-1}(r) - \varphi^{-1}(s),$$

which proves that $\varphi^{-1}$ is $\oplus$-convex.

Now suppose $\varphi^{-1}$ is $\oplus$-convex. Let $u_1 \le u_2$ and $v$ be arbitrary. Define $r = \varphi(u_2)$, $s = \varphi(u_1)$, and $t = \varphi(v)$. Then (3) implies (5), which proves that $C$ is monotonic by Lemma 8. □

The following simple theorem shows that every associative copula can be obtained in the way described in Theorem 7.

**Theorem 9.** *For every associative copula $C$ there exist $\oplus$ and $\varphi$ as in Theorem 7, with $a = 1$, such that*

$$C(x, y) = \varphi^{-1} (\varphi(x) \oplus \varphi(y)) .$$

Proof: It suffices to define

$$r \oplus s = 1 - C(1 - r, 1 - s)$$

and

$$\varphi(t) = 1 - t.$$

□

The copula defined by (2) is commutative, because $+$ is a commutative operation. Since, in Theorem 7, commutativity plays no role, one may expect that by using a noncommutative $\oplus$ we can construct noncommutative associative copulas. However, this is not possible. In [1] the following theorem is proved.

**Theorem 10.** *Let $T : I^2 \to I$ be a continuous mapping such that*

$$T(x, 0) = T(0, y) = 0 \text{ and } T(x, 1) = T(1, x) = x \text{ for all } x \in I,$$

*and*

$$T(T(x, y), z) = T(x, T(y, z)) \text{ for all } x, y, z \in I.$$

*Then*

$$T(x, y) = T(y, x) \text{ for all } x \in I.$$

From this result we easily obtain the following property of the operation $\oplus$.

**Theorem 11.** *A continuous associative operation $\oplus$ in $[0, a]$, $a \in [0, \infty]$, such that $t \oplus 0 = 0 \oplus t = t$ and $t \oplus a = a \oplus t = a$ for all $t \in [0, a]$, is commutative.*

Proof: If $a < \infty$, then define

$$T(x, y) = 1 - \frac{1}{a} ((a - ax) \oplus (a - ay)) .$$

If $a = \infty$, then define

$$T(x, y) = \frac{1}{1 + \frac{1-x}{x} \oplus \frac{1-y}{y}} .$$

In both cases $T$ satisfies the assumptions of Theorem 10, and thus $T(x, y) = T(y, x)$ for all $x, y \in I$. Now commutativity of $\oplus$ follows easily. □

Note that although every decreasing function is $\oplus$-convex with respect to the operation $\oplus$ defined in Example 3, we do not get a variety of copulas this way. Indeed,

$$\varphi^{-1} (\max(\varphi(x), \varphi(y))) = \min(x, y),$$

whenever $\varphi$ is decreasing.

## References

[1] Alsina C., Frank M.J., Schweizer B., *Associative functions on intervals*, in preparation.
[2] Nelsen R., *An Introduction to Copulas*, Springer-Verlag, Berlin-Heidelberg-New York-London-Paris-Tokyo-Hong Kong-Barcelona-Budapest, 1997.
[3] Schweizer B., Sklar A., *Probabilistic Metric Spaces*, North-Holland, New York-Amsterdam-Oxford, 1983.

Mathematics Department, Univ. of Central Florida, Orlando, FL 32816-1364, USA

(*Received* December 11, 1998, *revised* July 7, 1999)