Elliptic boundary value problem in Vanishing Mean Oscillation hypothesis

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Dedicated to the memory of Professor Filippo Chiarenza

Abstract. In this note the well-posedness of the Dirichlet problem (1.2) below is proved in the class $H_0^{1,p}(\Omega)$ for all $1 < p < \infty$ and, as a consequence, the Hölder regularity of the solution u.

$\mathcal{L}$ is an elliptic second order operator with discontinuous coefficients (VMO) and the lower order terms belong to suitable Lebesgue spaces.

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1. Introduction

Let us consider the Dirichlet problem for the equation

$$\mathcal{L}u + b_i u_{x_i} - (d_i u)_{x_i} + cu = (f_j)_{x_j} \tag{1.1}$$

in an open bounded set $\Omega \subset \mathbb{R}^n$, $n \geq 3$, where we assume $\mathcal{L}$ to be the elliptic second order operator in the divergence form

$$\mathcal{L} \equiv -\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right)$$

with discontinuous coefficients a_{ij} which belong to the Sarason class VMO of the vanishing mean oscillation functions (see [23]). VMO is the subspace of the John-Nirenberg's space BMO (see [14]) whose elements have norm on the balls vanishing as the radius of the ball approaches zero (see Section 2 for definitions). This hypothesis will be crucial to obtain our results. The lower order terms b_i, c, d_i belong to suitable Lebesgue spaces $L^s(\Omega)$.

The aim of this note is to prove the well-posedness of the following Dirichlet problem

$$\begin{cases} \mathcal{L}u + b_i u_{x_i} - (d_j u)_{x_j} + cu = (f_j)_{x_j} & \text{a.e. } x \in \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.2}$$

in the class of weak solutions $u \in H_0^{1,p}(\Omega)$ for all $1 < p < \infty$.

Then we extend the result contained in [8] in order to allow operators to have lower order terms.

In our treatment we will always assume the following

Hypothesis I.

$$\begin{cases} I_1 \;\; a_{ij}(x) \in VMO \cap L^\infty(\mathbb{R}^n) & \forall\, i,j = 1,\dots,n, \\ I_2 \;\; a_{ij}(x) = a_{ji}(x) & \forall\, i,j = 1,\dots,n, \text{ a.e. in } \Omega, \\ I_3 \;\; \exists\, \tau > 0 : \; \tau^{-1}|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \tau|\xi|^2, & \forall\, \xi \in \mathbb{R}^n, \text{ a.e. } \; x \in \Omega, \end{cases}$$

and

$$b_i, d_i \in L^r(\Omega) \;\forall\, i = 1,\dots,n \; \textit{with} \;\; \begin{cases} r = n \text{ if } \; 1 < p < n, \\ r > n \text{ if } p = n, \\ r = p \text{ if } p > n, \end{cases}$$

$c \in L^{\frac{r}{2}}(\Omega)$ where r is defined as above.

We also make the following assumption

$$c - (d_j)_{x_j} \ge c_0 > 0.$$

We next enunciate the main results of this note, while for the precise meaning of the hypothesis

$$a_{ij}(x) \in VMO, \quad \forall\, i,j = 1,\dots,n$$

we refer to Section 2.

Theorem 1.1. *Let a_{ij}, b_i, c, d_i verify Hypothesis I, $f \in [L^p(\Omega)]^n$, $1 < p < \infty$, and $\partial\Omega \in C^{1,1}$.*

Then the Dirichlet problem (1.2) *has a unique solution and there exists a constant k independent on u and f such that*

$$\|\nabla u\|_{L^p(\Omega)} \le k\|f\|_{L^p(\Omega)}.$$

Theorem 1.2. *Let a_{ij}, b_i, c, d_i satisfy Hypothesis I, $f \in [L^p(\Omega)]^n$, $p > n$ and $\partial\Omega \in C^{1,1}$.*

The solution of (1.2) *is Hölder regular in $\overline{\Omega}$ and there exists a constant k independent on u and f such that*

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} \le k\|f\|_{L^p(\Omega)}.$$

The hypothesis $a_{ij} \in VMO$ allows us to extend classical results obtained only for $p = 2$ with hypothesis $a_{ij} \in L^\infty$ (see e.g. [15], [12], [17]) to all $p \in]1, +\infty[$.

We also observe that the structure of the equation in the divergence form and the non existence of the derivatives of the coefficients a_{ij} leads us to examine the weak and not the strong solutions even if there is a certain similarity in the technique used to study both strong and weak solutions.

During this century the variational approach to the Dirichlet problem for linear elliptic equations has been object of much research and has been developed by many authors. Far from being complete we recall the research of Ladyzhenskaya, Uralt'seva, see [15], and Stampacchia, see [26] and [27]. These authors derived the Fredholm alternative but their existence and uniqueness results were restricted by smallness or coercivity conditions.

The Dirichlet problem was also considered by Friedrichs in [10], [11] and Garding in [13].

Furthermore we wish to mention classical results by Miranda, see [19] and [18] who deals with the case of strong solutions, with hypotheses $a_{ij} \in H^{1,n}$, $b_i, c \in L^n$, $d_i = 0$.

Higher order differentiability theorems for weak solutions were proved by various authors including Browder [2], Nirenberg in [21] and [22], Agmon in [1], Lax in [16], Bers and Schechter in [3] and Friedman in [9].

We also recall the celebrated paper [7] by De Giorgi in which the author studies local pointwise estimates. The global bound appears in the works of Ladyzhenskaya and Uralt'seva [15] and Stampacchia [26], [27] and is an extension of an earlier version by Stampacchia [24], [25]. A priori bound is due to Trudinger in [28].

The method used in this paper, following the idea of the papers [4], [5], is based on explicit representation formulas for the first derivatives. It permits us to obtain interior and boundary estimates for the solution of the Dirichlet problem (1.2) (respectively Lemma 3.1 and 3.2). In the interior case the integral operators appearing in the representation formula are Calderón-Zygmund singular integrals and singular commutators like those used by Coifman, Rochberg and Weiss in [6].

The boundary estimates are similar because the representation formula obtained using the half space Green function contains the same integral operators as in the interior case and a second type which are less singular operators.

Finally, both interior and boundary estimates assuring the global regularity for the first derivatives of a solution of (1.2) are used to prove in Theorem 1.1 the well-posedness of (1.2). As a consequence the Hölder regularity of u is proved in Theorem 1.2.

2. Definitions and preliminary results

Definition 2.1 (see [14]). *We say that a function $f \in L^1_{loc}(\mathbb{R}^n)$ belongs to the space BMO if*

$$\sup_B \frac{1}{|B|} \int_B |f(x) - f_B|\, dx \equiv \|f\|_* < \infty$$

where B is a ball in $\mathbb{R}^n$ and f_B is the average $\frac{1}{|B|}\int_B f(x)\, dx$.

BMO is a Banach space with the norm $\|f\|_*$ modulo constant functions, see [20].

Let $f \in BMO$ and $r > 0$. We set

$$\eta(r) = \sup_{\substack{x\in\mathbb{R}^n \\ \rho\le r}} \frac{1}{|B_\rho|} \int\limits_{B_\rho} |f(x) - f_{B_\rho}|\, dx$$

where B_ρ is a ball of radius ρ centered at the point $x \in \mathbb{R}^n$.

Definition 2.2. *We say that a function $f \in BMO$ is in the space VMO if*

$$\lim_{r\to 0^+} \eta(r) = 0$$

and we call η the VMO modulus of the function f.

In the following we denote by η_{ij} the VMO modulus of a_{ij}, $i, j = 1, \dots, n$, and let $\|a\|_* = \sum_{i,j=1}^n \eta_{ij}$.

Definition 2.3. *Let $k : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$. We say that $k(x)$ is a Calderón-Zygmund kernel (C-Z kernel) if*

$k \in C^\infty(\mathbb{R}^n \setminus \{0\})$;
$k(x)$ is homogeneous of degree $-n$;
$\int_\Sigma k(x)\, dx = 0$, where $\Sigma = \{ x \in \mathbb{R}^n : |x| = 1\}$.

Definition 2.4. *We set*

$$\Gamma(x,\zeta) = \frac{1}{n(2-n)\omega_n \sqrt{\det\{a_{ij}(x)\}}} \left(\sum_{i,j=1}^n A_{ij}(x)\zeta_i\zeta_j \right)^{(2-n)/2}$$

for a.a. x and $\forall\, \zeta \in \mathbb{R}^n \setminus \{0\}$, where $A_{ij}(x)$ stand for the entries of the inverse matrix of the matrix $\{a_{ij}(x)\}_{i,j=1,\dots,n}$, and ω_n is the measure of the unit ball in $\mathbb{R}^n$. Also we denote

$$\Gamma_i(x,\zeta) = \frac{\partial}{\partial\zeta_i}\Gamma(x,\zeta), \quad \Gamma_{ij}(x,\zeta) = \frac{\partial}{\partial\zeta_i\partial\zeta_j}\Gamma(x,\zeta).$$

It is well known that $\Gamma_{ij}(x,\zeta)$ are Calderón-Zygmund kernels in the ζ variable.

Theorem 2.5 (see [4, Theorem 2.10])**.** *Let $k : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$ be such that*

(i) *$k(x,.)$ is a Calderón-Zygmund kernel for a.a. $x \in \mathbb{R}^n$;*
(ii) $\max_{|j|\le 2n} \left\| \frac{\partial^j}{\partial z^j} k(x,z) \right\|_{L^\infty(\mathbb{R}^n\times\Sigma)} = M < +\infty.$

Let also $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, $a \in L^\infty(\mathbb{R}^n)$.

For any $\varepsilon > 0$ and $x \in \mathbb{R}^n$ we set

$$K_\varepsilon f(x) = \int\limits_{|x-y|>\varepsilon} k(x, x-y) f(y)\, dy,$$

$$C_\varepsilon(a,f)(x) = \int\limits_{|x-y|>\varepsilon} k(x,x-y)(a(x)-a(y))f(y)\,dy.$$

Then, there exist Kf, $C(a,f) \in L^p(\mathbb{R}^n)$ such that

$$\lim_{\varepsilon\to 0} \|K_\varepsilon f - Kf\|_{L^p(\mathbb{R}^n)} = 0, \qquad \lim_{\varepsilon\to 0} \|C_\varepsilon(a,f) - C(a,f)\|_{L^p(\mathbb{R}^n)} = 0$$

and there exists a constant $c = c(n,p,M)$ such that

$$\|Kf\|_{L^p(\mathbb{R}^n)} \le c\|f\|_{L^p(\mathbb{R}^n)}, \quad \|C(a,f)\|_{L^p(\mathbb{R}^n)} \le c\|a\|_*\|f\|_{L^p(\mathbb{R}^n)}.$$

As in [4] the functions Kf and $C(a,f)$ obtained by the above limiting process are called *Principal Value* functions and the notations usually used to indicate that Kf and $C(a,f)$ are such linear functionals, are

$$Kf(x) = P.V.\int\limits_{\mathbb{R}^n} k(x,x-y)f(y)\,dy$$

and

$$C(a,f)(x) = a(Kf) - K(af).$$

The result we are going to mention follows from the above theorem.

Theorem 2.6 (see [4, Theorem 2.13]). *Let $a \in VMO \cap L^\infty(\mathbb{R}^n)$ and $k(x,z)$ satisfy the hypothesis of Theorem 2.5. Then for any $\epsilon > 0$, there exists $\rho_0 > 0$ such that for any ball B_r of radius $r \in]0,\rho_0[$ and $f \in L^p(B_r)$ with $1 < p < \infty$ we have*

$$\|C(a,f)\|_{L^p(B_r)} \le c\,\epsilon\,\|f\|_{L^p(B_r)}.$$

Let us define $\mathbb{R}^n_+ = \{x = (x_1 \dots, x_n) \equiv (x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > 0\}$ and for $x \in \mathbb{R}^n$ let $\tilde{x} = (x', -x_n)$.

Analogous inequalities are proved in [5], as we recall in the next theorem, for the following operators

$$\tilde{K}f(x) = \int\limits_{\mathbb{R}^n_+} \frac{f(y)}{|\tilde{x}-y|^n}\,dy$$

and

$$\tilde{C}(a,f)(x) = \int\limits_{\mathbb{R}^n_+} \frac{[a(x)-a(y)]}{|\tilde{x}-y|^n} f(y)\,dy,$$

where $a \in VMO \cap L^\infty(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n_+)$, $1 < p < \infty$.

Theorem 2.7. *Let $f \in L^p(\mathbb{R}^n_+)$ with $1 < p < \infty$, and let $\tilde{K}f$ and $\tilde{C}(a,f)(x)$ be defined as above.*

Then there exists a constant c independent of f and ϕ such that

$$\|\tilde{K}f\|_{L^p(\mathbb{R}^n_+)} \leq c\|f\|_{L^p(\mathbb{R}^n_+)}$$

and

$$\|\tilde{C}(a,f)\|_{L^p(\mathbb{R}^n_+)} \leq c\|a\|_*\|f\|_{L^p(\mathbb{R}^n_+)}.$$

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be an open bounded domain with $\partial\Omega \in C^{1,1}$. Consider in Ω the elliptic equation (1.1) or, equivalently,

$$\mathcal{L}u = (f_j + d_j u)_{x_j} - (b_i u_{x_i} + cu) \tag{2.1}$$

and the associated Dirichlet problem

$$\begin{cases} \mathcal{L}u = (f_j + d_j u)_{x_j} - (b_i u_{x_i} + cu), \\ u \in H_0^{1,p}(\Omega), \qquad 1 < p < \infty. \end{cases} \tag{2.2}$$

In our treatment we assume that $f = (f_1, \dots, f_n) \in [L^p(\Omega)]^n$ with $1 < p < \infty$.

We shall say that $u \in H_0^{1,p}(\Omega)$, $1 < p < \infty$, is a *weak solution* of the Dirichlet problem (1.2) if

$$\int_\Omega (a_{ij} u_{x_i} \phi_{x_j} - b_i u_{x_i}\phi - cu\phi)\, dx = -\int_\Omega (f_j + d_j u)\phi_{x_j}\, dx, \quad \forall\, \phi \in C_0^\infty(\Omega). \tag{2.3}$$

3. Proofs of Theorems 1.1 and 1.2

Now we shall make some preliminary observations.

Let θ be a standard cut-off function, $\theta \in C_0^\infty(\mathbb{R})$, such that for fixed $r \in \mathbb{R}$ and every $s : 0 < s < r$

$$\theta(x) = \begin{cases} 1 & x \in B_s, \\ 0 & x \notin B_r. \end{cases}$$

Then if u is a solution of (1.2) we have

$$\begin{aligned} \mathcal{L}(\theta u) &= -\big(a_{ij}(\theta u)_{x_i}\big)_{x_j} \\ &= \mathcal{L}(\theta u) - \theta\mathcal{L}u + \theta\{(f_j + d_j u)_{x_j} - (b_i u_{x_i} + cu)\} \\ &= -\big(a_{ij}(\theta_{x_i} u + \theta u_{x_i})\big)_{x_j} - \theta\{-(a_{ij}u_{x_i})_{x_j}\} + \theta\{(f_j + d_j u)_{x_j} - b_i u_{x_i} + cu\} \\ &= -(a_{ij}\theta_{x_i} u - \theta(f_j + d_j u))_{x_j} - (a_{ij}\theta_{x_j} u_{x_i} + \theta_{x_j}(f_j + d_j u) + \theta b_i u_{x_i} + c\theta u). \end{aligned}$$

Then we write, for $v = \theta u$,

$$\mathcal{L}(v) \equiv \mathcal{L}(\theta u) = \operatorname{div}(\Phi) + \Psi \tag{3.1}$$

with Φ, Ψ supported in B_r and defined by

$$\Phi \equiv -(a_{ij}\theta_{x_i}u - \theta(f_j + d_j u))$$

and

$$\Psi \equiv -(a_{ij}\theta_{x_j}u_{x_i} + \theta_{x_j}(f_j + d_j u) + \theta b_i u_{x_i} + c\theta u).$$

In the following we consider only $p > 2$ because the case $p = 2$ is classical and $1 < p < 2$ will be obtained by duality.

Before proving Theorem 1.1 and Theorem 1.2 we need the following two lemmas.

Lemma 3.1. *Let $u \in C^\infty(\Omega)$ such that* (2.3) *is satisfied, let θ and v be defined as above.*

Let also $a_{ij} \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, such that I_2 and I_3 of Hypothesis I are true. Let also $f \in [C^\infty(\Omega)]^n$ and $b_i, d_i, c \in C^\infty(\Omega)$, for every $i, j = 1, \dots, n$.

Then there exist $r > 0$ and $C = C(n, p, \tau, \eta_{ij}, \mathrm{dist}(B_r, \partial\Omega))$ such that

$$\|\nabla u\|_{L^p(B_s)} \leq C\left(\|\nabla u\|_{L^2(B_r)} + \|f\|_{L^p(B_r)} + \|u\|_{L^p(B_r)}\right) \tag{3.2}$$

for every $s \in]0, r[$.

Proof: Let us define

$$\theta(x) = \begin{cases} 1 & x \in B_{\rho r}, \ 0 < \rho < 1, \\ 0 & x \notin B_r. \end{cases}$$

Set $\mathcal{L}(v) = \mathrm{div}(\Phi) + \Psi$. If $a_{ij} \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $\Phi \in [C_0^\infty(B_r)]^n$, $\Psi \in C_0^\infty(B_r)$ we have (see [8]) the representation formula and the consequent estimate based on Theorem 2.5 and Theorem 2.6

$$\begin{aligned} v_{x_i}(x) = \mathrm{P.V.} &\int_{B_r} \Gamma_{ij}(x, x-y)\{(a_{kj}(x) - a_{kj}(y))v_{x_k}(y) - \Phi_j(y)\}\, dy \\ &+ c_{ij}\Phi_j(x) - \int_{B_r} \Psi(y)\Gamma_i(x, x-y)\, dy, \ \ \forall\, x \in B_r \end{aligned}$$

with $c_{ij} = \int_{|\xi|=1} \Gamma_i(x, \xi)\xi_j d\sigma_\xi$,

$$\|\nabla v\|_{L^p(B_r)} \leq C\left(\|a\|_* \|\nabla v\|_{L^p(B_r)} + \|\Phi\|_{L^p(B_r)} + \|\Psi\|_{L^{p_*}(B_r)}\right) \tag{3.3}$$

where $C \geq 0$ does not depend on v, Φ, Ψ and p_* such that $\frac{1}{p_*} = \frac{1}{p} + \frac{1}{n}$.

Fixing $r > 0$ so small that $C\|a\|_*$ is less than 1 it follows

$$\|\nabla v\|_{L^p(B_r)} \leq C\left(\|\Phi\|_{L^p(B_r)} + \|\Psi\|_{L^{p_*}(B_r)}\right). \tag{3.4}$$

From (3.4) we have

$$\|\nabla(\theta u)\|_{L^p(B_r)} \leq C\Big(\|a_{ij}\theta_{x_i}u - \theta(f_j + d_j u)\|_{L^p(B_r)} \\ +\|a_{ij}\theta_{x_j}u_{x_i} + \theta_{x_i}(f_j + d_j u) + \theta(b_i u_{x_i} + cu)\|_{L^{p*}(B_r)}\Big)$$

and then

$$\|\nabla(\theta u)\|_{L^p(B_r)} \leq C\Big(\|a_{ij}\theta_{x_i}u\|_{L^p(B_r)} + \|\theta f_j\|_{L^p(B_r)} + \|\theta d_j u\|_{L^p(B_r)} \\ + \|a_{ij}\theta_{x_j}u_{x_i}\|_{L^{p*}(B_r)} + \|\theta_{x_i} f_j\|_{L^{p*}(B_r)} \\ +\|\theta_{x_i} d_j u\|_{L^{p*}(B_r)} + \|\theta b_i u_{x_i}\|_{L^{p*}(B_r)} + \|c\theta u\|_{L^{p*}(B_r)}\Big).$$

Let us suppose at the beginning $2 < p \leq 2^*$ where 2^* is such that $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{n}$; then $p_* \leq 2$.

Majorizing each term we have

$$\|a_{ij}\theta_{x_i}u\|_{L^p(B_r)} \leq C_1\|u\|_{L^p(B_r)},$$

$$\|\theta f_j\|_{L^p(B_r)} + \|\theta d_j u\|_{L^p(B_r)} \leq \|f\|_{L^p(B_r)} + \|d_j\|_{L^r(B_r)}\|\theta u\|_{L^{p_{\mathcal{S}}}(B_r)} \\ \leq \|f\|_{L^p(B_r)} + \mathcal{S}\|d_j\|_{L^r(B_r)}\|\nabla(\theta u)\|_{L^p(B_r)}$$

where $p_{\mathcal{S}} = \frac{pn}{n-p}$ and $\mathcal{S}$ is Sobolev constant,

$$\|a_{ij}\theta_{x_j}u_{x_i}\|_{L^{p*}(B_r)} \leq C_2\|\nabla u\|_{L^{p*}(B_r)} \leq C_2\|\nabla u\|_{L^2(B_r)}$$
$$\|\theta_{x_i} f_j\|_{L^{p*}(B_r)} \leq C_3\|f\|_{L^p(B_r)}$$

and, using Hölder inequality,

$$\|\theta_{x_i} d_j u\|_{L^{p*}(B_r)} \leq \|d_j\|_{L^r(B_r)}\|u\|_{L^p(B_r)}.$$

Moreover,

$$\|\theta b_i u_{x_i}\|_{L^{p*}(B_r)} = \|b_i[(\theta u)_{x_i} - \theta_{x_i}u]\|_{L^{p*}(B_r)} \\ \leq \|b_i(\theta u)_{x_i}\|_{L^{p*}(B_r)} + \|b_i\theta_{x_i}u\|_{L^{p*}(B_r)} \\ \leq \|b_i\|_{L^r(B_r)}\|\nabla(\theta u)\|_{L^p(B_r)} + C_4\|b_i\|_{L^r(B_r)}\|u\|_{L^p(B_r)},$$

$$\|c\theta u\|_{L^{p*}(B_r)} \leq \|c\|_{L^{\frac{r}{2}}(B_r)}\|\theta u\|_{L^{p^*}(B_r)} \leq \mathcal{S}\|c\|_{L^{\frac{r}{2}}(B_r)}\|\nabla(\theta u)\|_{L^p(B_r)}.$$

Fix $\tilde{r} > 0$ so small that

$$\Big[\|b_i\|_{L^r(B_r)} + \mathcal{S}\left(\|d_j\|_{L^r(B_r)} + \|c\|_{L^{\frac{r}{2}}(B_r)}\right)\Big] < \frac{1}{3C},$$

then for every $r \in]0, \tilde{r}]$ we have proved

$$\begin{aligned} \|\nabla u\|_{L^p(B_s)} &\leq \|\nabla(\theta u)\|_{L^p(B_r)} \\ &\leq C\left(\|u\|_{L^p(B_r)} + \|f\|_{L^p(B_r)} + \|\nabla u\|_{L^{p_*}(B_r)}\right), \quad \forall\, s \in]0, r[\ . \end{aligned} \tag{3.5}$$

Let us now prove (3.2) if $2 < p \leq 2^*$, choosing

$$\theta(x) = \begin{cases} 1 & x \in B_{\rho r}, \ \ 0 < \rho < 1, \\ 0 & x \notin B_r. \end{cases}$$

From (3.5) we obtain

$$\begin{aligned} \|\nabla u\|_{L^p(B_{\rho r})} &\leq C\left(\|u\|_{L^p(B_r)} + \|f\|_{L^p(B_r)} + \|\nabla u\|_{L^{p_*}(B_r)}\right) \\ &\leq C\left(\|u\|_{L^p(B_r)} + \|f\|_{L^p(B_r)} + \|\nabla u\|_{L^2(B_r)}\right) \end{aligned} \tag{3.6}$$

because $p_* \leq 2$, and then we get (3.2) choosing $\rho = \frac{s}{r}$.

Let us define 2^{**} such that $\frac{1}{2^{**}} = \frac{1}{2^*} - \frac{1}{n}$.

Set $2^* < p \leq 2^{**}$ (observe that we put formally $2^{**} = \infty$ and take $2^* < p < \infty$ provided $2^* \geq n$); then $p_* \leq 2^*$, and

$$\theta(x) = \begin{cases} 1 & x \in B_{\rho^2 r}, \ \ 0 < \rho < 1, \\ 0 & x \notin B_{\rho r}. \end{cases}$$

Using again (3.4) we have

$$\begin{aligned} \|\nabla(\theta u)\|_{L^p(B_{\rho^2 r})} &\leq C\left(\|u\|_{L^p(B_{\rho r})} + \|f\|_{L^p(B_{\rho r})} + \|\nabla u\|_{L^{p_*}(B_{\rho r})} + \|f\|_{L^{p_*}(B_{\rho r})}\right) \\ &\leq C\left(\|u\|_{L^p(B_r)} + \|f\|_{L^p(B_r)} + \|\nabla u\|_{L^{2^*}(B_{\rho r})}\right) \end{aligned}$$

and, majorizing the last term with (3.6) for $p = 2^*$,

$$\|\nabla u\|_{L^p(B_{\rho^2 r})} \leq C\left(\|u\|_{L^p(B_r)} + \|f\|_{L^p(B_r)} + \|\nabla u\|_{L^2(B_r)}\right).$$

We obtain again (3.2) choosing $\rho = \left(\frac{s}{r}\right)^{\frac{1}{2}}$.

Finally the estimate (3.2) is obtained for every $p > 2$ iterating this method a finite number of times. More precisely it is always possible to get $m \in \mathbb{N}$ such that $p_{m-1} < p \leq p_m$ with $p_{m-1} = 2^{\overbrace{**\cdots*}^{m-1}}$, $p_m = 2^{\overbrace{**\cdots*}^{m}}$, then setting $\rho = \left(\frac{s}{r}\right)^{\frac{1}{m}}$ the result is obtained.

The technique used here is similar to that in [8]. $\square$

Let us define $B_r^+ = \{x = (x_1, \ldots, x_n) \equiv (x', x_n) \in B_r : x_n > 0\}$.

Lemma 3.2. *There exists a positive number r such that if*

(i) $a_{ij}(x) \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $\forall\, i,j = 1,\dots,n$ *such that* I_2 *and* I_3 *are true;*

(ii) $u \in C^\infty(B_r^+)$ *is a solution of* (1.2) *in* B_r^+, u *vanishing on* $\{x_n = 0\} \cap \overline{B}_r^+$;

(iii) $f \in [C_0^\infty(\overline{B}_r^+)]^n$;

(iv) $b_i, c, d_i \in C^\infty(B_r^+)$, $\qquad \forall\, i = 1,\dots,n$;

then

$$\|\nabla u\|_{L^p(B_s^+)} \le C \left(\|u\|_{L^p(B_r^+)} + \|f\|_{L^p(B_r^+)} + \|\nabla u\|_{L^2(B_r^+)} \right)$$

where $C = C(n, p, \tau, \eta_{ij}, \operatorname{dist}(B_r^+, \partial\Omega))$.

Proof: Let $\theta \in C_0^\infty(B_r^+)$ and let $v = \theta u$ be a solution of (3.1). It is easy to show that the representation formula for the first derivatives of v is

$$\begin{aligned} v_{x_i}(x) = {} & \text{P.V.} \int_{B_r^+} \Gamma_{ij}(x, x-y) \left\{ (a_{kj}(x) - a_{kj}(y)) v_{x_h}(y) \right. \\ & \left. -(a_{ij}\theta_{x_i} u - \theta(f_j + d_j u))_j(y) \right\} dy \\ & + \int_{B_\gamma^+} (a_{ij}\theta_{x_j} u_{x_i} + \theta_{x_j}(f_j + d_j u) + \theta b_i u_{x_i} + c\theta u)(y) \Gamma_i(x, x-y)\, dy \\ & + c_{ij}(a_{ij}\theta_{x_i} u - \theta(f_j + d_j u))_j(x) + I_i(x), \quad \forall\, x \in B_r^+, \end{aligned}$$

where c_{ij} is defined as above and

$$\begin{aligned} I_i(x) = {} & \int_{B_r^+} \Gamma_{ij}(x, T(x) - y) \left\{ (a_{kj}(x) - a_{kj}(y)) v_{x_k}(y) \right. \\ & \left. -(a_{ij}\theta_{x_i} u - \theta(f_j + d_j u))_j(y) \right\} dy, \quad \text{for } 1 \le i < n; \\ I_n(x) = {} & \int_{B_r^+} \Gamma_{kj}(x, T(x) - y) A_k(x) \left\{ (a_{kj}(x) - a_{kj}(y)) v_{x_h}(y) \right. \\ & \left. -(a_{ij}\theta_{x_i} u - \theta(f_j + d_j u))_j(y) \right\} dy, \end{aligned}$$

where $A(y) = (A_1(y), \dots, A_n(y)) = T(e_n, y) \equiv T((0, \dots, 0, 1), y)$ and T is defined by

$$T(x, y) = x - \frac{2x_n}{a_{nn}(y)} \boldsymbol{a}_n(y), \qquad T(x) \equiv T(x, x),$$

and $\boldsymbol{a}_n(y) = (\, a_{in}(y)\,)_{i=1,\dots,n}$ is the last row (column) of the matrix $\boldsymbol{a}(y) = \{a_{ij}(y)\}_{i,j=1,\dots,n}$.

We also have, using Theorem 2.7, that there exists a positive number $r > 0$ and a positive constant $\overline{C}$ such that

$$\|\nabla v\|_{L^p(B_r^+)} \le \overline{C} \left(\|\Phi\|_{L^p(B_r^+)} + \|\Psi\|_{L^{p^*}(B_r^+)} \right),$$

where $\overline{C}$ is independent of the functions v, Φ and Ψ. Then similarly to Lemma 3.1 we get the conclusion. $\square$

We are now ready to establish the main result of the paper.

Proof of Theorem 1.1

We first observe that it is possible to find subsequences $\{(a_{ij})_h\}_{h\in\mathbb{N}}$, $\{(b_i)_h\}_{h\in\mathbb{N}}$, $\{c_h\}_{h\in\mathbb{N}}$, $\{(d_i)_h\}_{h\in\mathbb{N}}$, $\{f_h\}_{h\in\mathbb{N}}$, with $(a_{ij})_h \in C^\infty(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, $f_h \in [C^\infty(\Omega)]^n$, $(b_i)_h, c_h, (d_i)_h \in C^\infty(\Omega)$, $\forall\, i,j = 1,\dots,n$, such that $\{(a_{ij})_h\}$ converges in the $*-$norm to a_{ij}, $\{f_h\}$ converges to f in $[L^p(\Omega)]^n$ and $\{(b_i)_h\}$, $\{c_h\}$, $\{(d_i)_h\}$ are respectively converging to b_i, c, d_i in $L^r(\Omega)$, $\forall\, i = 1,\dots,n$.

We first prove the theorem with smooth hypothesis on the coefficients and the known term, then in the second step with the assumption requested.

FIRST STEP.

Let $a_{ij} \in C^\infty(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, $f \in [C^\infty(\Omega)]^n$, $b_i, c, d_i \in C^\infty(\Omega)$, $\forall\, i,j = 1,\dots,n$.

From Lemma 3.1 and Lemma 3.2 by a covering and flattering argument (see [5, Theorem 4.2])

$$\|\nabla u\|_{L^p(\Omega)} \le C\left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \|\nabla u\|_{L^2(\Omega)}\right), \quad \forall\, p > 2. \tag{3.7}$$

Let $2 < p \le 2^*$ $(p_* \le 2)$. From Sobolev theorem

$$\|u\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^{p_*}(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)}.$$

Then by (3.7) and the well known L^2-results obtained by Miranda (see [18])

$$\begin{aligned}\|\nabla u\|_{L^p(\Omega)} &\le C\left(\|\nabla u\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)}\right)\\ &\le C\left(\|f\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)}\right) \le C\|f\|_{L^p(\Omega)}.\end{aligned} \tag{3.8}$$

Let us suppose now $2^* < p \le 2^{**}$; then it follows $p_* \le 2^*$. From Sobolev theorem

$$\|u\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^{p_*}(\Omega)} \le C\|\nabla u\|_{L^{2^*}(\Omega)}. \tag{3.9}$$

Applying (3.8) with $p = 2^*$, from (3.9) we have

$$\|u\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^{2^*}(\Omega)} \le C\|f\|_{L^{2^*}(\Omega)} \le C\|f\|_{L^p(\Omega)}.$$

Then using the above inequality, (3.7) and the L^2-results mentioned above, we obtain

$$\|\nabla u\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}, \quad \text{for } \; p \le 2^{**}.$$

The last inequality for every $p > 2$ can be obtained iterating this method.

SECOND STEP.

Let us consider the above sequences of smooth functions; and u_h, $\forall\, h \in \mathbb{N}$, the solution of the associated Dirichlet problem.

Then there exists a constant C independent of h such that

$$\|\nabla u_h\|_{L^p(\Omega)} \leq C\|f_h\|_{L^p(\Omega)}, \quad \forall\, h \in \mathbb{N}.$$

Using the above inequality we have that $\exists u \in H_0^{1,p}(\Omega)$ verifying

$$\|\nabla u\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}$$

where u is the solution of (1.2).

This completes the proof of Theorem 1.1 with the constant $k = k(n, p, \tau, \eta_{ij}, \partial\Omega)$. □

Proof of Theorem 1.2

It is easy to see that it is a consequence of Theorem 1.1 and of the Sobolev imbedding theorem. □

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References

[1] Agmon S., *Lectures on Elliptic boundary Value Problems*, Van Nostrand, N.J., 1965.

[2] Browder F.E., *Strongly elliptic systems of differential equations*, in: Contributions to the Theory of Partial Differential Equations, Princeton University Press, Princeton N.J., 1954, pp. 15–51.

[3] Bers L., Schechter M., *Elliptic Equations In Partial Differential Equations*, Interscience, New York, 1964, pp. 131–299.

[4] Chiarenza F., Frasca M., Longo P., *Interior $W^{2,p}$ estimates for non-divergence elliptic equations with discontinuous coefficients*, Ricerche Mat. **40** (1991), 149–168.

[5] Chiarenza F., Frasca M., Longo P., *$W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc. **336** (1993), 841–853.

[6] Coifman R., Rochberg R., Weiss G., *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (1976), 611–635.

[7] De Giorgi E., *Sulla differenziabilita' e l'analiticita' delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. **3** (1957), 25–43.

[8] Di Fazio G., *L^p Estimates for divergence form elliptic equations with discontinuous coefficients*, Boll. U.M.I. (7) **10-A** (1996), 409–420.

[9] Friedman A., *Partial Differential Equations*, Holt, Rinehart and Winston, New York, 1969.

[10] Friedrichs K.O., *The identity of weak and strong extensions of differential operators*, Trans. Amer. Math. Soc. **55** (1944), 132–151.

[11] Friedrichs K.O., *On the differentiability of the solutions of linear elliptic equations*, Comm. Pure Appl. Math. **6** (1953), 299–326.

[12] Gilbarg D., Trudinger N.S., *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer-Verlag, 1983.
[13] Garding L., *Dirichlet's problem for linear elliptic partial differential equations*, Math. Scand. **1** (1953), 55–72.
[14] John F., Nirenberg L., *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426.
[15] Ladyzenskaya O.A., Uraltśeva N.N., *Linear and Quasilinear Elliptic Equations*, English Translation: Academic Press, New York, 2nd Russian ed., 1973.
[16] Lax P.D., *On Cauchy's problem for hyperbolic equations and the differentiability of solutions of elliptic equations*, Comm. Pure Appl. Math. **8** (1955), 615–633.
[17] Miranda C., *Partial Differential Equations of Elliptic Type*, second revised edition, Springer Verlag, 1970.
[18] Miranda C., *Sulle equazioni ellittiche del secondo ordine di tipo non variazionale, a coefficienti discontinui*, Annali di Matematica **63** (1963), 353–386.
[19] Miranda C., *Alcune osservazioni sulla maggiorazione in L^ν delle soluzioni deboli delle equazioni ellittiche del secondo ordine*, Annali di Matematica **61** (1963), 151–170.
[20] Neri U., *Some properties of bounded mean oscillation*, Studia Math. **41** (1977), 63–75.
[21] Nirenberg L., *On nonlinear elliptic partial differential equations and Hölder continuity*, Comm. Pure Appl. Math. **6** (1953), 103–156.
[22] Nirenberg L., *Remarks on strongly elliptic partial differential equations*, Comm. Pure Appl. Math. **5** (1955), 649–675.
[23] Sarason D., *On functions of vanishing mean oscillation*, Trans. Amer. Math. Soc. **207** (1975), 391–405.
[24] Stampacchia G., *Contributi alla regolarizzazione delle soluzioni dei problemi al contorno per equazioni del secondo ordine ellittiche*, Ann. Scuola Norm. Sup. Pisa (3) **12** (1958), 223–245.
[25] Spampacchia G., *I problemi al contorno per le equazioni differentiali di tipo ellittico*, Atti VI Congr. Un. Mat. Ital., Napoli Edizioni Cremonese, 1960, pp. 21–44.
[26] Stampacchia G., *Le probléme de Dirichlet pour les équations elliptiques du second ordre á coefficients discontinuous*, Ann. Inst. Fourier **15** (1965), 189–258.
[27] Stampacchia G., *Equations elliptiques du second ordre á coefficients discontinues*, Séminaire de Mathématiques Supérieures **16**, Les Presses de l'Université de Montréal, 1966.
[28] Trudinger N.S., *Linear elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa (3) **27** (1973), 265–308.

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