On reductive and distributive algebras

Anna Romanowska

Abstract. The paper investigates idempotent, reductive, and distributive groupoids, and more generally Ω-algebras of any type including the structure of such groupoids as reducts. In particular, any such algebra can be built up from algebras with a left zero groupoid operation. It is also shown that any two varieties of left k-step reductive Ω-algebras, and of right n-step reductive Ω-algebras, are independent for any positive integers k and n. This gives a structural description of algebras in the join of these two varieties.

Keywords: idempotent and distributive groupoids and algebras, Mal'cev products of varieties of algebras, independent varieties

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Introduction

The paper investigates the structure of algebras generalizing certain idempotent and distributive groupoids. Such groupoids are algebras $(A,\cdot)$ with one binary operation satisfying the idempotent and distributive laws:

$$x\cdot x = x, \tag{I}$$

$$x\cdot(y\cdot z)=(x\cdot y)\cdot(x\cdot z),\quad \text{and } (x\cdot y)\cdot z=(x\cdot z)\cdot(y\cdot z). \tag{D}$$

A systematic study of such groupoids was undertaken by Ježek, Kepka and Němec [JKN] in 1981. Much more recent notes of Dehornoy [De] show that such groupoids are really interesting algebras, with a rich theory and many applications. The groupoids we are interested in here have an additional "reductive" property. Multiplying an element x by an element y certain number of times, either only on the left or only on the right, returns the element y.

The idempotent Ω-algebras (A,Ω) we are interested in also have a binary (term) operation $\cdot$ that makes $(A,\cdot)$ an idempotent and distributive groupoid. Moreover, they are distributive, i.e. the operation $\cdot$ distributes both from the left and the right over each Ω-operation. It is known ([PiR]) that in any such algebra (A,Ω), the operation $\cdot$ acts as a kind of "partition" operation, and allows a decomposition of (A,Ω) into a disjoint union of left-reductive subalgebras. On the other hand, the Mal'cev product of the varieties of left m-step reductive and of left n-step reductive algebras is contained in the class of $m+n$-step left reductive algebras.

A stronger result where these two classes coincide was obtained in [PiR] for the case of Ω-modes, i.e. idempotent and entropic Ω-algebras, satisfying the identities

$$x \dots x\omega = x, \tag{I}$$

$$\begin{aligned}(x_{11} \dots x_{1n}\omega) \dots (x_{m1} \dots x_{mn}\omega)\omega' \\ = (x_{11} \dots x_{m1}\omega') \dots (x_{1n} \dots x_{mn}\omega')\omega\end{aligned} \tag{E}$$

for each n-ary ω and m-ary ω' in Ω. Note that, in particular, modes are distributive algebras. In Section 1, we obtain a similar result for idempotent and distributive algebras in the case $n = 2$ or $n = 3$. This, together with results of [RT], gives a nice structural description of left 3-reductive algebras, and in particular of left 3-reductive idempotent and distributive groupoids.

The second part of the paper extends some other results of [PiR]. We show that the varieties of left k-step reductive and of right n-step reductive Ω-algebras are independent for any positive integers k and n. This result, together with the previous ones, gives a structural description of algebras in the join of the above varieties. The presence of entropicity again gives stronger results, and a much simpler proof of the independence. See [PiR]. The paper concludes with some comments and questions.

We use notation and terminology similar to that in the book [RS]. In particular, words (terms) and operations are denoted by $x_1 \dots x_n w$ instead of $w(x_1, \dots, x_n)$, with the exception of traditional binary operations. The symbol $x_1 \dots x_n w$ means that $x_1, \dots, x_n$ are exactly the variables appearing in the word w. For a congruence α of an algebra (A, Ω), the quotient algebra is denoted by (A^α, Ω), and for a in A, the α-class containing a is denoted by a^α.

1. Left and right reductive algebras

Throughout this paper let $r : \Omega \to \mathbb{N}$ be a fixed type of algebras and let $x \cdot y$ by a fixed Ω-word with precisely two variables x and y. Consider the following *n-step left reductive* (or briefly *n-reductive*) law

$$x^n y := x \cdot (x \cdot (\dots (x \cdot y) \dots)) = x. \tag{r_n}$$

In what follows we are interested in idempotent varieties of Ω-algebras satisfying the identity (r_n) for some positive integer n, and additionally the *left* and *right distributive* laws

$$x \cdot (x_1 \dots x_m\omega) = (x \cdot x_1) \dots (x \cdot x_m)\omega, \tag{ld}$$

$$(x_1 \dots x_m\omega) \cdot x = (x_1 \cdot x) \dots (x_m \cdot x)\omega \tag{rd}$$

for each (m-ary) ω in Ω. We denote such varieties by R_n and call them (*left*) *n-reductive varieties.* We refer to R_n-algebras as *n-reductive algebras.* An Ω-algebra is *left reductive* if it is n-reductive for some positive integer n.

An Ω-algebra is called *n-step right reductive* or briefly *right n-reductive* if it satisfies the *right n-reductive law*

$$(r'_n) \qquad yx^n := (\dots((y \cdot x) \cdot x)\dots)x = x$$

opposite to (r_n) and the left and right distributive laws (ld) and (rd). It is called *right reductive* if it is right n-reductive for some positive integer n. The right n-reductive variety is denoted by R'_n. Each fact we formulate for left reductive algebras may easily be reformulated in the opposite way for right reductive algebras.

Left n-reductive varieties may be easily obtained from idempotent irregular varieties. Let V be an idempotent irregular variety of Ω-algebras, i.e. a variety satisfying an identity with different sets of variables on each side. Such a variety is known to have a basis for its identities consisting of the set Σ of regular identities true in V and an identity of the form

$$(i) \qquad x \cdot y = x.$$

(See e.g. [PłR].) In other words, the variety V is strongly irregular ([PłP]), and as is easily seen, 1-reductive. The set Σ of regular identities true in V defines the *regularization* $\widetilde{V}$ of V ([PłR]). Evidently $\widetilde{V}$ contains V, so $\widetilde{V}$ is a supervariety of V. Other supervarieties of V, interesting for us in this note, are the varieties $R_n(V)$ defined by the idempotent laws, the distributive laws (ld) and (rd) obviously true for V, and the n-reduction law (r_n). Note that the varieties $R_n(V)$ are all contained in the idempotent variety $D(V)$ of Ω-algebras defined by the identities (ld) and (rd). Note also that the variety $R_n(V)$ depends on the term $x \cdot y$ chosen for the axiomatization of the variety V.

In general, consider for a fixed Ω-word $x \cdot y$, the idempotent variety V defined by the above distributive laws (ld) and (rd). Let U and W be subvarieties of V. Recall that the *Mal'cev product* $U \circ W$ of U and W (relative to V) consists of V-algebras (A, Ω) with a congruence θ such that (A^θ, Ω) is in W, and each θ-class (a^θ, Ω) is in U. The product $U \circ W$ is a quasivariety ([M]), but in general it is not a variety. The rôle of Mal'cev products for n-reductive varieties is explained by the following.

Theorem 1.1 ([PiR]). *Let V be the idempotent variety of Ω-algebras defined by all the left distributive laws (ld). Let n be a positive integer. Then all k-reductive subvarieties $R_k(V)$ of V, for $k < n$, are related as follows:*

$$R_{n-k}(V) \circ R_k(V) \subseteq R_n(V). \qquad \square$$

A better result is obtained in the case of mode varieties, i.e. idempotent varieties satisfying the entropic laws. Note that the idempotent and entropic laws imply all distributive laws (ld) and (rd) for each (derived) binary operation.

Theorem 1.2 ([PiR]). *Let V be the variety of Ω-modes. Let n be a positive integer. Then all k-reductive subvarieties $R_k(V)$ of V, for $k < n$, are related as follows:*

$$R_{n-k}(V) \circ R_k(V) = R_n(V). \qquad \square$$

In particular, Theorem 1.2 implies that $\circ$ is associative and commutative, and

$$R_n(V) = (R_1(V))^n.$$

The paper [RT] provides some construction methods for $R_n(V)$-algebras from $R_{n-k}(V)$-and $R_k(V)$-algebras.

The proof of Theorem 1.2 (see [PiR]) is based on the following:

Lemma 1.3 ([PRR], [PiR]). *For a fixed type $r : \Omega \to \mathbb{Z}^+$ and an Ω-term $x \cdot y$, the following identities are equivalent in the variety of Ω-modes*

(i) $$x^n y = x,$$

(ii) $$x_1 \cdot (x_2 \cdot \ldots (x_{n-1} \cdot x_n) \ldots) = x_1 \cdot (x_2 \cdot \ldots (x_n \cdot y) \ldots) \qquad \square$$

Lemma 1.3 remains true for $n = 2$ and $n = 3$, if one drops entropicity, and instead assumes both distributive laws (ld) and (rd). Let D be the idempotent variety of Ω-algebras satisfying the distributive laws (ld) and (rd).

Lemma 1.4. *Let $x \cdot y$ be an Ω-term as above. Then the following two identities are equivalent in the variety D:*

(i) $$x^2 y = x,$$

(ii) $$x \cdot yz = xy.$$

Proof: The implication (ii) $\Rightarrow$ (i) is obvious. We will prove (i) $\Rightarrow$ (ii). Applying repeatedly 2-reductive and distributive laws, one gets the following

$$\begin{aligned} x \cdot yz &= (x^2 y)(yz) \\ &= (x \cdot yz)(xy \cdot yz) \\ &= (xy \cdot xz)(xy \cdot yz) \\ &= xy \cdot (xz \cdot yz) \\ &= xy \cdot (xy \cdot z) \\ &= xy. \end{aligned} \qquad \square$$

Lemma 1.5. *For an Ω-term as above, the following two identities are equivalent in the variety D:*

(i) $$x^3 y = x,$$

(ii) $$x(y \cdot zt) = x \cdot yz.$$

Proof: The implication (ii) $\Rightarrow$ (i) is obvious. We will prove (i) $\Rightarrow$ (ii). First we show several consequences of the identity (i), if it holds in the variety D:

(a) $$x(y \cdot xt) = xy \cdot x.$$

Applying distributivity and (i) one obtains:

$$\begin{aligned} x(y \cdot xt) &= xy \cdot x^2 t = x^3 t \cdot (y \cdot x^2 t) \\ &= x(y \cdot x^2 t) = xy \cdot x^3 t = xy \cdot x. \end{aligned}$$

(b) $$x^2(zt) = x^2(zx).$$

We again use distributivity, (i) and (a) to show the following:

$$\begin{aligned} x^2(zt) &= x^2 2 \cdot x^2 t = x^3 t \cdot (xz \cdot x^2 t) \\ &= x \cdot (xz \cdot x^2 t) = x(x(z \cdot xt)) \\ &= x(xz \cdot x) = x^2(zx). \end{aligned}$$

Now (b) obviously implies

(c) $$x^2 z = x^2(zt).$$

(d) $$x(y^2 t) = xy.$$

This identity follows by distributivity, and the identities (a) and (c):

$$\begin{aligned} x(y^2 t) &= xy \cdot (x \cdot yt) = x^2(yt) \cdot y(x \cdot yt) \\ &= x^2(yt) \cdot yxy = x^2 y \cdot yxy \\ &= xy \cdot xy = xy. \end{aligned}$$

Now we are ready to prove that (i) implies (ii):

$$\begin{aligned} x(y \cdot zt) &= xy \cdot (x \cdot zt) \\ &= x(x \cdot zt) \cdot y(x \cdot zt) \\ &= x^2 z \cdot y(x \cdot zt) && \text{by } (c) \end{aligned}$$

$$\begin{aligned}
&= (x \cdot y(x \cdot zt)) \cdot (xz \cdot y(x \cdot zt)) && \text{by } (a)\\
&= (xy \cdot x)((xy \cdot x)(z \cdot y(x \cdot zt)))\\
&= (xy \cdot x)((xy \cdot x)z) && \text{by } (c)\\
&= (xy \cdot x)((xy \cdot z) \cdot xz) = ((xy \cdot x)(xy \cdot z))((xy \cdot x)(xz))\\
&= (xy \cdot xz)(xyx \cdot xz) = (xy \cdot xyx) \cdot xz\\
&= x(y^2x \cdot z) = x(y^2x) \cdot xz\\
&= xy \cdot xz = x \cdot yz. && \text{by } (d) \qquad \square
\end{aligned}$$

Lemmata 1.4 and 1.5 make it possible to use the same proof as for Theorem 1.2 in the following situation.

Theorem 1.6. *The left reductive subvarieties of the subvariety $R_3(D)$ of D are related as follows:*

$$\begin{aligned}
R_2(D) &= R_1(D) \circ R_1(D),\\
R_3(D) &= R_1(D) \circ R_2(D) = R_2(D) \circ R_1(D)\\
&= R_1(D) \circ (R_1(D) \circ R_1(D))\\
&= (R_1(D) \circ R_1(D)) \circ R_1(D). \qquad \square
\end{aligned}$$

In particular, if D is the variety IDG of idempotent and distributive groupoids, i.e. groupoids satisfying the distributive laws

$$x \cdot yz = xy \cdot xz \quad \text{and} \quad xy \cdot z = xz \cdot yz,$$

then $R_1(D) = Lz$, the variety of left-zero semigroups. In this case one can write:

$$\begin{aligned}
R_2(D) &= (Lz)^2,\\
R_3(D) &= (Lz)^3.
\end{aligned}$$

In general, we do not know whether the inclusion in Theorem 1.1 can be replaced with equality as in Theorem 1.2.

In subsequent sections we will be interested in the relation between general left k-reductive and right n-reductive varieties of Ω-algebras defined for a fixed binary word $x \cdot y$.

2. Independent joins of varieties

Let V_1 and V_2 be varieties of Ω-algebras of the same fixed type. The varieties V_1 and V_2 are *independent* if there is an Ω-word x_1x_2d with two variables x_1 and x_2, called a *decomposition word*, such that the identity $x_1x_2d = x_i$ holds in V_i for $i = 1, 2$. It is well known that whenever the varieties V_1 and V_2 are independent, each algebra (A, Ω) in their join $V = V_1 \vee V_2$ is isomorphic to a product $(A_1, \Omega) \times (A_2, \Omega)$, with (A_i, Ω) in V_i for $i = 1, 2$, and the algebras (A_i, Ω) are determined up to isomorphism. In this case, we denote the join V of V_i by $V_1 + V_2$ and say that V is an *independent join* of the subvarieties V_1 and V_2. (See [GLP]. Note however that V is called a "product" there. "Direct sum" is another name used for such a join [RS].) As was shown in [Kn], in the case where the independent varieties V_1 and V_2 have finite bases for their identities, their join $V_1 + V_2$ is also finitely based. In the case where V_1 is a left reductive variety, and V_2 is a right reductive variety, it is very easy to find the basis for $V_1 + V_2$.

Proposition 2.1. *Let V_1 and V_2 be varieties of Ω-algebras, the first one being k-reductive and the second one right n-reductive for a fixed Ω-word $x \cdot y$. If V_1 and V_2 are independent, and x_1x_2d is a corresponding decomposition word, then the independent join $V = V_1 + V_2$ is the idempotent variety of algebras satisfying all the distributive identities (ld) and (rd), and additionally the following ones:*

$$
\begin{aligned}
&x_{11}x_{12}dx_{21}x_{22}dd = x_{11}x_{22}d,\\
&(x_{11}\dots x_{1m}\omega)(x_{21}\dots x_{2m}\omega)d = (x_{11}x_{21}d)\dots(x_{1m}x_{2m}d)\omega,\\
&(x^k y)zd = xzd,\\
&x(yz^n)d = xzd
\end{aligned}
$$

for each (m-ary) ω in Ω. □

The proof goes exactly as the proof of Proposition 3.2 in [PRR], where a similar result is formulated for mode varieties. We will omit it here.

3. On the independence of left and right reductive varieties

In this section, it will be shown that for a fixed Ω-word $x \cdot y$ as in Section 1, and any positive numbers k and n, the varieties R_k of k-reductive Ω-algebras and R'_n of right n-reductive Ω-algebras are independent.

For a fixed n, we define a sequence of binary Ω-words as follows.

$$
\begin{aligned}
&d_1 := xy^n,\\
&d_2 := xd_1^n = x(xy^n)^n, \dots,\\
&d_{m+1} := xd_m^n.
\end{aligned}
$$

In what follows, D will denote the idempotent supervariety of R_k and R'_n defined by all the distributive laws (ld) and (rd). We start with a number of lemmas that will eventually show that the words d_k are decomposition words for the varieties R_k and R'_n.

Lemma 3.1. *The variety D satisfies the following identities for each positive m:*

$$xd_{m+1} = x(xd_m)^n = (x^2d_m)(xd_m)^{n-1}.$$

Proof: By definition

$$\begin{aligned} xd_{m+1} &= x(xd_m^n) \\ &= x((xd_m)d_m^{n-1}) \\ &= (x^2d_m)(xd_m)^{n-1} \qquad \text{(by distributivity)} \\ &= x(xd_m)^n. \end{aligned}$$

□

Lemma 3.2. *The variety D satisfies the following identity for each $m \geq 2$:*

$$\begin{aligned} xd_m =&((\dots(((x^m d_1)(x^{m-1}d_1)^{n-1})((x^{m-1}d_1)(x^{m-2}d_1)^{n-1})^{n-1}) \ \dots) \\ &(\dots \quad ((x^3d_1)(x^2d_1)^{n-1})^{n-1}\dots)^{n-1}) \\ &((\dots(((x^{m-1}d_1)(x^{m-2}d_1)^{n-1})((x^{m-2}d_1)(x^{m-3}d_1)^{n-1})^{n-1}) \ \dots) \\ &(\dots \quad ((x^2d_1)(xd_1)^{n-1})^{n-1}\dots)^{n-1})^{n-1}. \end{aligned}$$

Proof: By induction on m. For $m = 2$, Lemma 3.1 implies that

$$xd_2 = x(xd_1)^n = (x^2d_1)(xd_1)^{n-1}.$$

To make the calculations in the general case more readable, let us calculate xd_3, too:

$$\begin{aligned} xd_3 &= x(xd_2)^n = (x^2d_2)(xd_2)^{n-1} \\ &= x((x^2d_1)(xd_1)^{n-1})((x^2d_1)(xd_1)^{n-1})^{n-1} \\ &= ((x^3d_1)(x^2d_1)^{n-1})((x^2d_1)(xd_1)^{n-1})^{n-1}, \end{aligned}$$

the first and second equalities following by Lemma 3.1, and the fourth by distributivity.

To make the notation and calculations easier, we introduce a certain encoding of the expressions appearing in xd_m. For $i = 1, \dots, m$, we denote by i the expression $x^i d_1$, and we replace by j any power $n - j$. Thus

$$\begin{aligned} &x^m d_1 =: m \\ &(x^m d_1)(x^{m-1}d_1)^{n-1} =: m(m-1)^1. \end{aligned}$$

The word xd_m is encoded as

$$\begin{aligned} xd_m =&((\dots((m(m-1)^1)((m-1)(m-2)^1)^1) \quad \dots) \\ &(\dots \quad (32^1)^1\dots)^1) \\ &((\dots(((m-1)(m-2)^1)((m-2)(m-3)^1)^1) \quad \dots) \\ &(\dots \quad (21^1)^1\dots)^1)^1. \end{aligned}$$

We will show that if the identity of Lemma 3.2 holds for m, then it also holds for $m+1$. Again, we use Lemma 3.1 and the distributivity. So assume that the identity holds for m. Then since by distributivity

$$\begin{aligned} x(ab^j) &= x(ab^{j-1}) \cdot (xb) \\ &= (x(ab^{j-2}) \cdot (xb)) \cdot (xb) \\ &= \quad \dots \\ &= (xa)(xb)^j, \end{aligned}$$

it follows that

$$\begin{aligned} xd_{m+1} &= (x^2 d_m)(xd_m)^{n-1} \\ &= [x \cdot xd_m][xd_m]^{n-1} \\ =&[((\dots(((m+1)m^1)(m(m-1)^1)^1) \qquad \dots) \\ &\quad (\dots \quad (43^1)^1 \dots)^1) \\ &((\dots((m(m-1)^1)((m-1)(m-2)^1)^1) \quad \dots) \\ &\quad (\dots \quad (21^1)^1 \dots)^1)^1] \\ &[((\dots((m(m-1)^1)((m-1)(m-2)^1)^1) \quad \dots) \\ &\quad (\dots \quad (32^1)^1 \dots)^1) \\ &((\dots(((m-1)(m-2)^1)((m-2)(m-3)^1)^1) \quad \dots) \\ &\quad (\dots \quad (21^1)^1 \dots)^1)^1]^1. \end{aligned}$$

By induction the identity of Lemma 3.2 holds for each $m \geq 2$. $\square$

Lemma 3.3. *The variety D satisfies the following identity for each positive p:*

$$x^p d_1 = (x^{p+1}y)(x^p y)^{n-1}.$$

Proof: By induction on p. For $p = 1$, distributivity implies

$$\begin{aligned} xd_1 = x(xy^n) &= x((xy)y^{n-1}) \\ &= (x^2 y)(xy)^{n-1}. \end{aligned}$$

Suppose now that the identity of 3.3 holds for p. Then distributivity implies

$$\begin{aligned} x^{p+1}d_1 = x(x^p d_1) &= x[(x^{p+1}y)(x^p y)^{n-1}] \\ &= (x^{p+2}y)(x^{p+1}y)^{n-1}. \end{aligned}$$

By induction, the identity of 3.3 holds for all positive p. $\square$

Lemma 3.4. *If a D-algebra (A, Ω) is $m+1$-reductive, then it satisfies the identity*

$$xd_m = x.$$

Proof: First note that Lemma 3.3 and the $m+1$-reductive law imply that

$$\begin{aligned} x^m d_1 &= (x^{m+1}y)(x^m y)^{n-1} \\ &= x\ (x^m y)^{n-1} \\ &= (x(x^m y))(x^m y)^{n-2} \\ &= x(x^m y)^{n-2} \\ &= \dots \\ &= x^{m+1}y = x. \end{aligned}$$

We introduce the following notation for subwords of xd_m:

$$b_1 := m-1,$$

$$b_2 := (m-1)(m-2)^1,$$

$$\dots$$

$$b_i := (\dots((m-1)(m-2)^1)\ \dots)(\dots\quad ((m-(i-1))(m-i)^1)^1 \dots)^1,$$

with $i-1$ powers 1 at the end, and

$$\begin{aligned} a_0 &:= m, \\ a_1 &:= mb_1^1, \\ a_2 &:= mb_2^1, \\ &\dots \\ a_i &:= mb_i^1, \end{aligned}$$

where $i = 1, 2, \dots, m-1$. We will show by finite induction on i that each a_i equals m. We know already that

$$a_0 = m.$$

Now

$$\begin{aligned} mb_1 = m(m-1) &= (x^m d_1)(x^{m-1}d_1) \\ &= x(x^{m-1}d_1) = x^m d_1 =: m, \end{aligned}$$

whence

$$\begin{aligned} a_1 = mb_1^1 = m(m-1)^1 &= (m(m-1))(m-1)^2 \\ &= m(m-1)^2 = (m(m-1))(m-1)^3 \\ &= m(m-1)^3 = \dots = m(m-1) = m. \end{aligned}$$

Now suppose that all $a_0, a_1, \dots, a_{i-1}$ equal m. Then

$$\begin{aligned}
mb_i &= m[(\dots(((m-1)(m-2)^1)((m-2)(m-3)^1)^1) \qquad \dots) \\
&\qquad (\dots \quad ((m-i+1)(m-i)^1)^1 \dots)^1] \\
&= \quad (\dots((m(m-1)^1)((m-1)(m-2)^1)^1) \qquad \dots) \\
&\quad (\dots \qquad ((m-i+2)(m-i+1)^1)^1)\dots)^1 \\
&= \qquad (\dots(a_1((m-1)(m-2)^1)^1) \qquad \dots) \\
&\quad (\dots \qquad ((m-i+2)(m-i+1)^1)^1 \dots)^1 \\
&= \qquad (\dots((mb_2^1)(((m-1)(m-2)^1)((m-2)(m-3)^1)^1)^1) \quad \dots) \\
&\quad (\dots \qquad ((m-i+2)(m-i+1)^1)^1 \dots)^1 \\
&= \qquad (\dots(a_2b_3^1) \qquad \dots) \\
&\quad (\dots \qquad ((m-i+2)(m-i+1)^1)^1 \dots)^1 \\
&= \quad \dots \\
&= m(\dots \qquad ((m-i+2)(m-i+1)^1)^1 \dots)^1 \\
&= mb_{i-1}^1 = a_{i-1} = m.
\end{aligned}$$

Hence

$$\begin{aligned}
a_i &= mb_i^1 = (mb_i)b_i^2 = mb_i^2 \\
&= (mb_i)b_i^3 = mb_i^3 = \dots \\
&= mb_i = m.
\end{aligned}$$

Since $a_0 = m = x^m d_1 = x$, it follows easily that $a_0 = a_1 = \dots = a_{m-1} = x$. Then Lemma 3.2 and the $m+1$-reductive law imply that

$$xd_m = mb_{m-1}^1 = a_{m-1} = mm = x. \qquad \square$$

Lemma 3.5. *If a D-algebra (A, Ω) is $m+1$-reductive, then it satisfies the identity*

$$d_{m+1} = x.$$

PROOF: By Lemma 3.4

$$\begin{aligned}
d_{m+1} = xd_m^n &= (xd_m)d_m^{n-1} = xd_m^{n-1} \\
&= (xd_m)d_m^{n-2} = xd_m^{n-2} = \dots = xd_m = x. \qquad \square
\end{aligned}$$

Lemma 3.6. *If a D-algebra (A, Ω) is right n-reductive, then it satisfies the identity $d_m = y$ for each positive number m.*

Proof: By the right n-reductive law

$$d_1 = xy^n = y.$$

Hence

$$\begin{aligned} d_2 &= xd_1^n = xy^n = y, \\ d_3 &= xd_2^n = xy^n = y, \\ &\dots, \\ d_m &= xd_{m-1}^n = xy^n = y. \end{aligned}$$

□

Theorem 3.7. *For a fixed Ω-word $x \cdot y$, and any positive numbers k and n, the varieties R_k of k-reductive Ω-algebras and R'_n of right n-reductive Ω-algebras are independent.*

Proof: Lemmas 3.1–3.6 give the proof. The word d_k is the decomposition word. □

Corollary 3.8. *The join of the varieties R_k and R'_n is independent, i.e.*

$$R_k \vee R'_n = R_k + R'_n.$$

□

Note that the right-distributive laws (rd) were not used in the proof of Theorem 3.7. However, assuming them allows us to use the dual version of Theorem 1.1 for R'_n-algebras, and thus makes it possible to describe the structure of $R_k + R'_n$-algebras.

4. Further comments and questions

If the varieties R_k and R'_n of the previous section are entropic, i.e. they are varieties of modes, the results of [PiR] show not only that R_k and R'_n are independent, but also that

$$R_k + R'_n = R_k \circ_E R'_n = R_{k,n}.$$

Here $\circ_E$ denotes the Mal'cev product relative to the variety of Ω-modes, and $R_{k,n}$ is the variety of Ω-modes defined by the identity

$$(r_{k,n}) \qquad x^k y x^n = x.$$

Note that the variety D satisfies the identity

$$x^k(yx_n) = (x^k y)x^n.$$

Since the variety of Ω-modes is a subvariety of the variety D, one can safely use notation as in $(r_{n,k})$. For reductive varieties we obviously have the following inclusions:

$$R_k + R'_n \subseteq R_k \circ R'_n \subseteq R_{k,n}.$$

Here the Mal'cev product is taken relative to the variety D, and $R_{k,n}$ is the subvariety of D defined by the identity $(r_{k,n})$. In the case $k = n = 1$, and D being the variety IDG of groupoids, it is well known that the following holds:

$$\begin{aligned} R_1 \circ R'_1 &= R_1 \circ_E R'_1 = Re = R_{1,1} \\ &= R_1 + R'_1 = Lz + Rz, \end{aligned}$$

where Re is the variety of rectangular semigroups and Rz is the variety of right zero semigroups. (See e.g. [Du]). In general, we do not know if the three classes $R_k + R'_n$, $R_k \circ R'_n$ and $R_{k,n}$ coincide. A positive solution of this problem, and of that at the end of Section 1, would give a nice characterization of the varieties $R_{k,n}$. Note also that for k and n equal 2 or 3, and D equal IDG, Theorem 1.6 implies that

$$R_k + R'_n = (Lz)^k + (Rz)^n.$$

The structure of groupoids in $(Lz)^k$ and in $(Rz)^n$ may be described using results of [RT].

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Warsaw Technical University, Institute of Mathematics, Plac Politechniki, 00661 Warsaw, Poland

and

Iowa State University, Department of Mathematics, Ames, Iowa 50011, USA

E-mail: aroman@alpha.im.pw.edu.pl

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