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Pointwise and locally uniform convergence of holomorphic and harmonic functions

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Abstract. We shall characterize the sets of locally uniform convergence of pointwise convergent sequences. Results obtained for sequences of holomorphic functions by Hartogs and Rosenthal in 1928 will be generalized for many other sheaves of functions. In particular, our Hartogs-Rosenthal type theorem holds for the sheaf of solutions to the second order elliptic PDE's as well as it has applications to the theory of harmonic spaces.

Keywords: Osgood's theorem, approximation, maximum principle, harmonic space, elliptic PDE's

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Introduction

In 1901, W.F. Osgood proved the theorem (known as Osgood's theorem) that a pointwise convergent sequence of holomorphic functions on a domain in $\mathbb{C}$ converges locally uniformly on an open dense subset of this domain (see [Osg]), leaving open the question what the set of locally uniform convergence looks like. A characterization of this set came several years later — F. Hartogs and A. Rosenthal (see [HR]) introduced special sets called *Streifen* and showed that, for any bounded domain $U \subset \mathbb{C}$ with connected complement and for any $V \subset U$ such that $V = \bigcup_{j\in J} V_j$ where V_j is a domain dense in U and $\mathbb{C} \setminus V_j$ is connected for any $j \in J$, the existence of a sequence of holomorphic functions which converges pointwise on U and locally uniformly exactly on V is equivalent to the existence of a family of *Streifen* possessing certain covering properties.

Among others, the family has to be pointwise finite and locally infinite on $U \setminus V$. Furthermore, they proved that this condition is equivalent to the existence of a family which is pointwise finite on U and locally infinite on $U \setminus V$. Analysing the proof, one can see that a family which is even locally finite on V is constructed. In this paper, we have chosen this last triad of covering properties and such a family will be called *admissible for* (U, V).

It turns out that it is not necessary to have such strong limitations on U and V. It is sufficient to require that the set U is open and its complement has no relatively compact connected component. Under these hypotheses it will be shown that a Hartogs-Rosenthal type theorem holds not only for holomorphic

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functions in $\mathbb{C}$, but for many other sheaves of functions in more general spaces. In particular, the case of solutions to the second order elliptic PDE's in $\mathbb{R}^d$ is included.

For this purpose it was necessary to modify the definition of *Streifen* (a domain enclosed by a polygon), and the term *führt aus U heraus* (another property used in [HR]) which requires of U to be bounded. Both problems were solved by introducing our *strips* instead of *Streifen*. In addition, in $\mathbb{C}$ the existence of a family of strips which is admissible for (U, V) is equivalent to the existence of a family of *Streifen* which possesses all covering properties from [HR].

The proof of the theorem consists of several mostly independent parts (an analogue to Runge's theorem, maximum principle etc.), each of them has its own paragraph in this paper. The last paragraph refers to applications and examples.

All lemmas are new, they generalize various parts of the proof of *Bedingung A* ([HR, p. 216-220]) using new terms such as *Runge sheaf*, *maximum property* etc.

As far as the holomorphic case is concerned, more historical notes and examples can be found in [U1] and [U2].

Runge sheaves

Let X be a topological space. (A topological space will, in this paper, always be Hausdorff.) For any subset A of X, we denote by A^o the interior of A. For any open subset U of X and any sequence $\{K_n\}_{n=1}^{\infty}$ of compact sets, the symbol $K_n \nearrow U$ means that $\bigcup_{n=1}^{\infty} K_n = U$ and $K_n \subset K_{n+1}^o$ for any $n \in \mathbb{N}$.

Let X be a topological space. A (real, resp. complex) *sheaf* on X is a map $\mathcal{G}$ defined on the set of non-empty open subsets of X such that:

(i) for any open non-empty subset U of X, $\mathcal{G}(U)$ is a real (resp. complex) vector space of real (resp. complex) functions on U,

(ii) for any two open sets U, V such that $\emptyset \neq U \subset V$ the restriction of any function from $\mathcal{G}(V)$ to U belongs to $\mathcal{G}(U)$,

(iii) for any non-empty open subset U of X and any covering $\{U_j\}_{j \in I}$ of U formed by non-empty open sets a function on U belongs to $\mathcal{G}(U)$ if for any $j \in I$ its restriction to U_j belongs to $\mathcal{G}(U_j)$.

A sheaf $\mathcal{G}$ on X is said to be a *continuous sheaf* if, for any non-empty open set U of X, the elements of $\mathcal{G}(U)$ are continuous on U. We shall say that a sheaf $\mathcal{G}$ on X is *non-degenerate at a point* $x \in X$ if there exist a neigbourhood U of x and an element f of $\mathcal{G}(U)$ such that $f(x) > 0$. A *non-degenerate sheaf* on X means, by definition, a sheaf which is non-degenerate at any point of X.

Let $\mathcal{G}$ be a sheaf on X, K be a compact subset of X. A pair $(\mathcal{G}, K)$ is said to possess the *Runge property* if, for any open set U such that $K \subset U \subset X$, for any $f \in \mathcal{G}(U)$ and any $\varepsilon > 0$, there exists $g \in \mathcal{G}(X)$ such that $|f - g| < \varepsilon$ on K. A compact subset K of X is called a *Runge set* if no connected component of $X \setminus K$ is relatively compact. A sheaf $\mathcal{G}$ on X is called a *Runge sheaf* if a pair $(\mathcal{G}, K)$ possesses the Runge property for each Runge set $K \subset X$.

Lemma 1. *Let X be a non-compact, locally compact, locally connected and connected space. Assume that, for any domain Z in X and any $x \in Z$, $Z \setminus \{x\}$ is also a domain. If $K \subset X$ is a Runge set and $y \notin K$, then $K \cup \{y\}$ is a Runge set.*

Proof: Since X is locally connected, there exists a domain U such that $y \in U$ and $U \cap K = \emptyset$. Since X is a connected Hausdorff space and $X \neq \{y\}$, the set $\{y\}$ is not open. Hence $U \setminus \{y\} \neq \emptyset$ and $y \in \overline{U \setminus \{y\}}$.
Let C be a connected component of $X \setminus (K \cup \{y\})$. Assume first that C contains the (non-empty connected) set $U \setminus \{y\}$. Since $y \in \overline{U \setminus \{y\}}$, $C \cup \{y\}$ is connected and therefore this set is a connected component of $X \setminus K$. Hence $C \cup \{y\}$ is not relatively compact. This means, C is not relatively compact.
On the contrary, assume that C does not contain $U \setminus \{y\}$. In this case, C is a connected component of $X \setminus K$. Hence C is not relatively compact. □

Let X be a topological space, $\{g_n\}_{n=1}^{\infty}$ be a sequence of functions on X. We denote by $PC(\{g_n\}_{n=1}^{\infty})$ the *set of pointwise convergence*, i.e., the set of all $z \in X$ such that $\{g_n\}_{n=1}^{\infty}$ converges at z. We denote by $CC(\{g_n\}_{n=1}^{\infty})$ the *set of compact convergence*, i.e., the set of all $z \in X$ for which there exists an open neigbourhood $U(z)$ such that $\{g_n\}_{n=1}^{\infty}$ converges uniformly on $U(z)$.

Lemma 2. *Let X be a locally compact, locally connected, non-compact space with countable base. Assume that there exists a sequence $\{K_n\}_{n=1}^{\infty}$ of Runge sets such that $K_n \nearrow X$. Let U be a non-empty open subset of X such that no connected component of $X \setminus U$ is relatively compact. Assume that $\mathcal{G}$ is a Runge sheaf on X. Then, for any sequence $\{f_n\}_{n=1}^{\infty}$ of elements of $\mathcal{G}(U)$, there exists a sequence $\{g_n\}_{n=1}^{\infty}$ of elements of $\mathcal{G}(X)$ such that*

$$\begin{aligned} PC(\{f_n\}_{n=1}^{\infty}) &= U \cap PC(\{g_n\}_{n=1}^{\infty}), \\ CC(\{f_n\}_{n=1}^{\infty}) &= U \cap CC(\{g_n\}_{n=1}^{\infty}). \end{aligned}$$

Proof: In case $U = X$ we take $g_n := f_n$ for any $n \in \mathbb{N}$.
Assume $U \neq X$. Let $x \in U$. Since $K_n \nearrow X$, there exists $m \in \mathbb{N}$ such that $x \in K_m^o$. Denote by M_i, $i \in I$, the connected components of $M := X \setminus U$. Since M is closed and $x \notin M$, there exists an open set V such that $M \subset V$ and $x \notin \overline{V}$. Denote by V_j, $j \in J$, the connected components of V, and define $J_1 := \{j \in J\ ;\ V_j \cap M \neq \emptyset\}$. Then the set $W := \bigcup_{j \in J_1} V_j$ is open, $M \subset W$ and $x \notin \overline{W}$. Define $R := K_m \setminus W$. Then $R \subset U$ and $x \in R^o$. We will show that R is a Runge set. Obviously, R is compact. Further, we have $X \setminus R = X \setminus (K_m \setminus W) = (X \setminus K_m) \cup W$. Since K_m is a Runge set, it is sufficient to show that no connected component of W is relatively compact. Let C be a connected component of W. Then there exists $j \in J_1$ such that $C = V_j$. It implies that $C \cap M \neq \emptyset$, i.e., there exists $i \in I$ such that $C \cap M_i \neq \emptyset$. Since $M_i \subset W$ is connected and C is a connected component of W, it follows that $M_i \subset C$. Since M_i is not relatively compact, C is not relatively compact.

Since $\mathcal{G}$ is a Runge sheaf, $(\mathcal{G}, R)$ possesses the Runge property. This means, for every $n \in \mathbb{N}$ there exists a function $g_n \in \mathcal{G}(X)$ such that $|f_n - g_n| < 2^{-n}$ on R. Taking $y \in R^o$ and $k, j \in \mathbb{N}$ we get

$$\begin{aligned}
|f_k(y) - f_j(y)| &\leq |f_k(y) - g_k(y)| + |g_k(y) - g_j(y)| + |g_j(y) - f_j(y)| \\
&< 2^{-k} + |g_k(y) - g_j(y)| + 2^{-j}, \\
|g_k(y) - g_j(y)| &\leq |g_k(y) - f_k(y)| + |f_k(y) - f_j(y)| + |f_j(y) - g_j(y)| \\
&< 2^{-k} + |f_k(y) - f_j(y)| + 2^{-j}.
\end{aligned}$$

Now it easily follows that $PC(\{f_n\}_{n=1}^{\infty}) = U \cap PC(\{g_n\}_{n=1}^{\infty})$ and $CC(\{f_n\}_{n=1}^{\infty}) = U \cap CC(\{g_n\}_{n=1}^{\infty})$. □

Maximum principle

Let X be a topological space, $\mathcal{G}$ be a sheaf on X. Then X is said to possess the *maximum property* with respect to $\mathcal{G}$ if

$$\sup \{ \, |g(w)| \; ; \; w \in U \, \} \; \leq \; \sup \{ \, \limsup_{y \to z} |g(y)| \; ; \; z \in \partial U \, \}$$

whenever $U \subset X$ is a non-empty relatively compact domain with non-empty boundary and $g \in \mathcal{G}(U)$.

Lemma 3. *Let X be a locally connected and non-compact space, $\mathcal{G}$ be a continuous sheaf on X. Assume that X possesses the maximum property with respect to $\mathcal{G}$. If g is an element of $\mathcal{G}(X)$ and $\alpha > 0$, then no connected component of $A := \{z \in X \; ; \; |g(z)| > \alpha\}$ is relatively compact.*

Proof: Let us assume that there exists a relatively compact connected component Z of A. Since X is locally connected and non-compact, Z is open and $\partial Z \neq \emptyset$. Since g is continuous on X and $|g| > \alpha$ on Z, we get $|g| \geq \alpha$ on ∂Z.
Assume first that there exists $z \in \partial Z$ such that $|g(z)| > \alpha$. It implies the existence of a connected neigbourhood U of z such that $|g| > \alpha$ on U. Then $Z \cup U$ is a connected subset of A which contains Z as a proper subset. We have a contradiction to the maximality of Z.
Therefore $|g| = \alpha$ on ∂Z. Since Z is a non-empty relatively compact domain, the maximum property gives us $|g| \leq \alpha$ on Z, which is a contradiction. □

Covering properties

Let X be a topological space, $\mathcal{A}$ be a family of subsets of X, $B \subset X$, $x \in X$. We say that $\mathcal{A}$ is *pointwise finite* at x if the set $\{A \in \mathcal{A} \; ; \; x \in A\}$ is finite. We say that $\mathcal{A}$ is *locally finite* at x if there exists a neigbourhood U of x such that the set $\{A \in \mathcal{A} \; ; \; U \cap A \neq \emptyset\}$ is finite. We say that $\mathcal{A}$ is *pointwise* (resp. *locally*) *infinite* if $\mathcal{A}$ is not pointwise (resp. locally) finite. We say that $\mathcal{A}$ is pointwise finite (resp. locally finite, pointwise infinite, locally infinite) on B, if $\mathcal{A}$ is pointwise finite (resp. locally finite, pointwise infinite, locally infinite) at any $x \in B$.

Remark 4. Let X be a topological space, $x \in X$. Let $\mathcal{R}$ be a family of subsets of X, $\mathcal{P}$ be a subfamily of $\mathcal{R}$. If $\mathcal{R}$ is pointwise (resp. locally) finite at x, then $\mathcal{P}$ is pointwise (resp. locally) finite at x.

Lemma 5. *Let X be a space with countable base, $\emptyset \neq F \subset X$. Let $\mathcal{R}$ be a family of subsets of X which is locally infinite on F. Then there exists a subfamily $\mathcal{P}$ of $\mathcal{R}$ which is countable and locally infinite on F.*

Proof: Let $\mathcal{U} = \{U_n \; ; \; n \in \mathbb{N}\}$ be a base of X. Define $\mathcal{U}' := \{U \in \mathcal{U} \; ; \; U \cap F \neq \emptyset\}$. For every $n \in \mathbb{N}$ we shall construct $\mathcal{P}_n$: If $U_n \notin \mathcal{U}'$, put $\mathcal{P}_n := \emptyset$. Otherwise, take $x \in U_n \cap F$. Thus we have $x \in F$ and a neigbourhood U_n of x. Since $\mathcal{R}$ is locally infinite on F, the set $\mathcal{R}_n := \{R \in \mathcal{R} \; ; \; R \cap U_n \neq \emptyset\}$ is infinite. Take $\mathcal{P}_n$ as an arbitrary countable infinite part of $\mathcal{R}_n$.
Putting $\mathcal{P} := \bigcup_{n=1}^{\infty} \mathcal{P}_n$ we get the desired subfamily of $\mathcal{R}$ which is countable and locally infinite on F. Indeed, for any $z \in F$ and any neigbourhood W of z there exists U_j, an element of base, such that $z \in U_j \subset W$. It follows immediately that $\mathcal{P}_j$ is infinite, which implies that $R \cap W \neq \emptyset$ for infinitely many $R \in \mathcal{P}$. □

Lemma 6. *Let X be a space with countable base. Let F be a non-empty subset of X. Let $\mathcal{R} := \{R_m \; ; \; m \in \mathbb{N}\}$ be a family of subsets of X. Assume that R_m is a closure of an open set for every $m \in \mathbb{N}$. If $\mathcal{R}$ is locally infinite on F, then there exist a subfamily $\mathcal{P} = \{P_n \; ; \; n \in \mathbb{N}\}$ of $\mathcal{R}$ and a sequence $\{b_n\}_{n=1}^{\infty}$ of points such that $b_n \in P_n^o$ for every $n \in \mathbb{N}$ and the set $\{n \in \mathbb{N} \; ; \; b_n \in W\}$ is infinite whenever $z \in F$ and W is a neigbourhood of z.*

Proof: Let $\mathcal{U}$ be a countable base of X. Define $\mathcal{U}' := \{U \in \mathcal{U} \; ; \; U \cap F \neq \emptyset\}$. Let $\mathcal{V} = \{V_n \; ; \; n \in \mathbb{N}\}$ be a family of sets which contains every element of $\mathcal{U}'$ infinitely many times, i.e., $V_n \in \mathcal{U}'$ and the set $\{k \; ; \; V_n = V_k\}$ is infinite for any $n \in \mathbb{N}$.
Since $\mathcal{R}$ is locally infinite on F, for any V_n there exist infinitely many elements of $\mathcal{R}$ which have non-empty intersection with V_n.
We shall construct P_n, $n \in \mathbb{N}$, by induction: Let P_1 be an arbitrary element of $\mathcal{R}$ such that $P_1 \cap V_1 \neq \emptyset$. Fix $n > 1$ and suppose that we have defined $P_1, \dots, P_{n-1}$. Let P_n be an element of $\mathcal{R}$ such that $P_n \neq P_j$ for any $j < n$ and $P_n \cap V_n \neq \emptyset$.
Define $\mathcal{P} := \{P_n \; ; \; n \in \mathbb{N}\}$.
Since P_n is a closure of an open set and $V_n \cap P_n \neq \emptyset$, we have $V_n \cap P_n^o \neq \emptyset$. Let b_n be a point of $V_n \cap P_n^o$.
Now let $z \in F$, W be a neigbourhood of z. Then there exists $U \in \mathcal{U}'$ such that $z \in U \subset W$. Since $U = V_n$ for infinitely many $n \in \mathbb{N}$, we have $b_n \in V_n = U \subset W$ for infinitely many $n \in \mathbb{N}$. This means, the set $\{n \in \mathbb{N} \; ; \; b_n \in W\}$ is infinite. □

Strips

A subset S of a topological space X is called a *strip* if S is the closure of a domain which is not relatively compact.
Let $V \subset U \subset X$, $\mathcal{S}$ be a family of subsets of X. Then $\mathcal{S}$ is said to be *admissible for* (U, V) if $\mathcal{S}$ is

(i) locally finite on V,

(ii) pointwise finite on $U \setminus V$,
(iii) locally infinite on $U \setminus V$.

Theorem 7. *Let X be a locally compact, locally connected, connected and non-compact space with countable base. Assume that there exists a sequence $\{K_n\}_{n=1}^{\infty}$ of Runge sets such that $K_n \nearrow X$. Moreover, assume that, for any domain Z in X and any $x \in Z$, $Z \setminus \{x\}$ is also a domain. Let $\mathcal{G}$ be a non-degenerate Runge sheaf on X. Let $V \subset U$ be non-empty open subsets of X. Suppose that there exists a family $\mathcal{S}$ of strips which is admissible for (U,V). Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of elements of $\mathcal{G}(X)$ such that*
(1) $\lim_{n\to\infty} f_n = 0$ *on* U,
(2) $U \cap CC(\{f_n\}_{n=1}^{\infty}) = V$.

Proof: Define $F := U \setminus V$. In case $F = \emptyset$ we take $f_n := 0$ for any $n \in \mathbb{N}$.
Assume $F \neq \emptyset$. Using Lemma 5 and Remark 4 we can suppose that $\mathcal{S}$ is countable. Denote by S_m, $m \in \mathbb{N}$, the elements of $\mathcal{S}$. Since $\mathcal{S}$ is locally infinite on F, by Lemma 6 there exist a subfamily $\mathcal{P} = \{P_n\ ;\ n \in \mathbb{N}\}$ of $\mathcal{S}$ and a sequence $\{b_n\}_{n=1}^{\infty}$ of points such that $b_n \in U \cap P_n^o$ for any $n \in \mathbb{N}$ and for any $z \in F$ and any neigbourhood W of z the set $\{n \in \mathbb{N}\ ;\ b_n \in W\}$ is infinite. It follows that the family $\{P_n^o\ ;\ n \in \mathbb{N}\}$ is locally infinite on F, thus the family $\mathcal{P}$ is locally infinite on F. Further, by Remark 4, $\mathcal{P}$ is admissible for (U,V).
Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of Runge sets such that $K_n \nearrow X$. Fix $n \in \mathbb{N}$ and define

$$R_n := K_n \ \cap\ (X \setminus P_n^o), \qquad T_n := R_n \ \cup\ \{b_n\}.$$

Obviously, R_n and T_n are compact. Further, we have $X \setminus R_n = (X \setminus K_n) \cup P_n^o$. Since K_n is a Runge set and P_n^o is not relatively compact, R_n is a Runge set. It follows by Lemma 1 that T_n is a Runge set.
Let V_n, W_n be open sets such that

$$\overline{V_n}\ \cap\ \overline{W_n}\ =\ \emptyset, \qquad R_n \subset V_n, \qquad b_n \in W_n.$$

Since $\mathcal{G}$ is non-degenerate at b_n, there are an open neigbourhood U_n of b_n, $U_n \subset W_n$, and a function $h_n \in \mathcal{G}(U_n)$ such that $h_n(b_n) > 0$.
Putting $Z_n := V_n \cup U_n$ and

$$g_n := \begin{cases} 0 & \text{on } V_n\ , \\ h_n/h_n(b_n) & \text{on } U_n\ , \end{cases}$$

we have $g_n|_{V_n} \in \mathcal{G}(V_n)$ and $g_n|_{U_n} \in \mathcal{G}(U_n)$, which implies $g_n \in \mathcal{G}(Z_n)$. The Runge property for T_n, Z_n, g_n and $\varepsilon = 2^{-n}$ yields a function $f_n \in \mathcal{G}(X)$ such that $|f_n - g_n| < 2^{-n}$ on T_n. This means,

$$|f_n(z)| < 2^{-n} \ \text{ for } \ z \in R_n, \qquad |f_n(b_n) - 1| < 2^{-n}.$$

Now we shall show that (1) and (2) hold for the sequence $\{f_n\}_{n=1}^{\infty}$.
Let $z \in F$. Since $\mathcal{P}$ is pointwise finite on F, there exists $m_0 \in \mathbb{N}$ such that $z \notin P_n$ for any $n > m_0$. Since $K_n \nearrow X$, there exists $m_1 > m_0$ such that $z \in K_n$ for any $n > m_1$. It implies $z \in R_n$ for any $n > m_1$. Using $|f_n| < 2^{-n}$ on R_n we get $\lim_{n\to\infty} f_n(z) = 0$.
Assume now that $z \in V$. Since $K_n \nearrow X$, there exists $m_2 \in \mathbb{N}$ such that $z \in K_n^o$ for any $n > m_2$. Since $\mathcal{P}$ is locally finite on V, there exist $m_3 > m_2$ and a neigbourhood $U(z)$ of z such that $U(z) \subset K_n^o$ and $U(z) \cap P_n = \emptyset$ for any $n > m_3$. This means that $U(z) \subset R_n$ for any $n > m_3$. Using $|f_n| < 2^{-n}$ on R_n we get $\lim_{n\to\infty} f_n(z) = 0$ and $z \in CC(\{f_n\}_{n=1}^{\infty})$.
It remains only to prove that $F \cap CC(\{f_n\}_{n=1}^{\infty}) = \emptyset$. Let $z \in F$ and let $U(z)$ be a neigbourhood of z. Fix $j \in \mathbb{N}$ and take $n > j$ such that $b_n \in U(z)$. Since $\lim_{m\to\infty} f_m(b_n) = 0$, we obtain $m > n$ such that $|f_m(b_n)| < \frac{1}{4}$. Further we have $|f_n(b_n) - 1| < 2^{-n}$, which implies $|f_n(b_n)| > \frac{1}{2}$. Therefore

$$|f_n(b_n) - f_m(b_n)| \;\geq\; |f_n(b_n)| - |f_m(b_n)| \;>\; \frac{1}{2} - \frac{1}{4} \;=\; \frac{1}{4}\,,$$

which means $z \notin CC(\{f_n\}_{n=1}^{\infty})$. □

The Vitali property

Let X be a topological space and $\mathcal{G}$ be a sheaf on X. The space X is said to possess the *Vitali property* with respect to $\mathcal{G}$ if, for any open subset U of X and for any locally uniformly bounded sequence $\{f_n\}_{n=1}^{\infty}$ of elements of $\mathcal{G}(U)$ with $PC(\{f_n\}_{n=1}^{\infty}) = U$, we have $PC(\{f_n\}_{n=1}^{\infty}) = CC(\{f_n\}_{n=1}^{\infty})$.

For Baire spaces with the Vitali property we have an Osgood type theorem:

Theorem 8. *Let X be a Baire space, $\mathcal{G}$ be a continuous sheaf on X. Assume X possesses the Vitali property with respect to $\mathcal{G}$. Suppose that U is an open subset of X, $\{f_n\}_{n=1}^{\infty}$ is a sequence of elements of $\mathcal{G}(U)$ such that $PC(\{f_n\}_{n=1}^{\infty}) = U$. Then $V := CC(\{f_n\}_{n=1}^{\infty})$ is open and dense in U.*

Proof: By definition, V is open. We shall show that V is dense in U. Take $z \in U$ and a neighbourhood W of z, $W \subset U$. Since $\{f_n\}_{n=1}^{\infty}$ converges pointwise on W, $\{f_n\}_{n=1}^{\infty}$ is pointwise bounded on W. This means that $W = \bigcup_{k=1}^{\infty} A_k$, where $A_k := \{z \in W \ ; \ |f_n| \leq k \text{ for every } n \in \mathbb{N}\}$. Since f_n is continuous for any $n \in \mathbb{N}$, the set A_k is closed for any $k \in \mathbb{N}$. By the hypothesis, X is a Baire space. It implies that there exists k_0 such that $W_1 := A_{k_0}^o$ is non-empty. Hence we get the non-empty open subset W_1 of W such that $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded on W_1. Using the Vitali property on W_1 we get that $\{f_n\}_{n=1}^{\infty}$ converges locally uniformly on W_1. This means, $W_1 \subset W \cap V$. Since W_1 is non-empty, the proof is complete. □

Theorem 9. *Let X be a locally compact, locally connected, connected and non-compact space with countable base. Let $\mathcal{G}$ be a continuous sheaf on X. Assume that X possesses the Vitali property and the maximum property with respect to $\mathcal{G}$. Let $V \subset U$ be open subsets of X. Suppose that there exists a sequence $\{f_m\}_{m=1}^{\infty}$ of elements of $\mathcal{G}(X)$ such that*

(1) $U \cap PC(\{f_m\}_{m=1}^{\infty}) = U$,
(2) $U \cap CC(\{f_m\}_{m=1}^{\infty}) = V$.

Then there exists a family $\mathcal{S}$ of strips which is admissible for (U, V).

Proof: Put $F := U \setminus V$. In case $F = \emptyset$ take $\mathcal{S} := \{X\}$. Assume $F \neq \emptyset$. Let $\mathcal{U}$ be a countable base of X. Define $\mathcal{U}' := \{W \in \mathcal{U}\ ;\ W \cap F \neq \emptyset\}$. Let $\mathcal{V} = \{V_n\ ;\ n \in \mathbb{N}\}$ be a family of sets which contains every element of $\mathcal{U}'$ infinitely many times, i.e., $V_n \in \mathcal{U}'$ and the set $\{k\ ;\ V_n = V_k\}$ is infinite for any $n \in \mathbb{N}$.
Fix $n \in \mathbb{N}$. Since $V_n \cap F \neq \emptyset$, by (2) the sequence $\{f_m\}_{m=1}^{\infty}$ does not converge locally uniformly on V_n. This means, by the Vitali property, that $\{f_m\}_{m=1}^{\infty}$ is not locally uniformly bounded on V_n. Since X is locally compact, there exists a relatively compact open set W_n such that $\overline{W_n} \subset V_n$. It follows that $\{f_m\}_{m=1}^{\infty}$ is not uniformly bounded on $\overline{W_n}$. Hence, there exist $k_n > n$ and $z_n \in \overline{W_n} \subset V_n$ such that $|f_{k_n}(z_n)| > n$. Define

$$A_n := \{z \in X\ ;\ |f_{k_n}(z)| > n\}.$$

Let Z_n be the connected component of A_n which contains z_n. Using Lemma 3 we get that Z_n is not relatively compact. Define

$$S_n := \overline{Z_n}.$$

It is easily seen that, for every $n \in \mathbb{N}$, $S_n \subset \{z \in X\ ;\ |f_{k_n}(z)| \geq n\}$ and S_n is a strip. It remains only to show that $\mathcal{S} := \{S_n\ ;\ n \in \mathbb{N}\}$ is admissible for (U, V).
Let $z \in V$. By (2), there exists a neigbourhood W of z such that $\{f_m\}_{m=1}^{\infty}$ converges uniformly on $\overline{W}$. We can assume that $\overline{W} \subset U$. Define $f := \lim_{m\to\infty} f_m$ on $\overline{W}$. Since f is continuous on the compact $\overline{W}$, there exists a strictly positive $M \in \mathbb{R}$ such that $|f| \leq M$ on $\overline{W}$. Since $\{f_m\}_{m=1}^{\infty}$ converges to f uniformly on $\overline{W}$, there exists $j > M$ such that $|f_m - f| < M$ on $\overline{W}$ for any $m > j$. Using $|f_m| \leq |f_m - f| + |f|$ we get that $|f_m| < 2M < 2j$ for any $m > j$. In particular, $|f_{k_m}| < 2j$ on W for any $m > j$. This means that for any $z \in V$ we have found W and j such that $W \cap S_n = \emptyset$ for any $n > 2j$, i.e., $\mathcal{S}$ is locally finite on V.
Let $z \in F$. By (1), we can define $f(z) := \lim_{m\to\infty} f_m(z)$. Define $M := |f(z)|$. Since $f(z) = \lim_{m\to\infty} f_m(z)$, there exists $j > M$ such that $|f_m(z) - f(z)| < M$ for any $m > j$. Using $|f_m| \leq |f_m - f| + |f|$ we get that $|f_m(z)| < 2M < 2j$ for any $m > j$. In particular, $|f_{k_m}(z)| < 2j$ for any $m > j$. This means that for any $z \in F$ we have found j such that $z \notin S_n$ for any $n > 2j$, i.e., $\mathcal{S}$ is pointwise finite on F.
It remains only to show that $\mathcal{S}$ is locally infinite on F. Let $z \in F$ and W be

a neigbourhood of z. Then there exists $V_n \in \mathcal{V}$ such that $z \in V_n \subset W$. Since $\{k\ ;\ V_k = V_n\}$ is infinite, we get that $z \in V_k \subset W$ for infinitely many k. Further, $z_k \in V_k \cap S_k$ for any k. Now it is easily seen that $W \cap S_k \neq \emptyset$ for infinitely many k. The proof is complete. $\square$

A Hartogs-Rosenthal type theorem

Theorem 10. *Let X be a locally compact, locally connected, connected and non-compact space with countable base. Assume that there exists a sequence $\{K_n\}_{n=1}^\infty$ of Runge sets such that $K_n \nearrow X$. Assume further that, for any domain Z in X and any $x \in Z$, $Z \setminus \{x\}$ is also a domain. Let $\mathcal{G}$ be a non-degenerate continuous Runge sheaf on X. Suppose that X possesses the Vitali property and the maximum property with respect to $\mathcal{G}$. Let $V \subset U$ be non-empty open subsets of X such that no connected component of $X \setminus U$ is relatively compact. Then the following assertions are equivalent:*

(A) *There exists a sequence $\{f_n\}_{n=1}^\infty$ of elements of $\mathcal{G}(U)$ such that*

  (1) $PC(\{f_n\}_{n=1}^\infty) = U$,
  (2) $CC(\{f_n\}_{n=1}^\infty) = V$.

(B) *There exists a sequence $\{f_n\}_{n=1}^\infty$ of elements of $\mathcal{G}(X)$ such that*

  (1) $U \cap PC(\{f_n\}_{n=1}^\infty) = U$,
  (2) $U \cap CC(\{f_n\}_{n=1}^\infty) = V$.

(C) *There exists a family $\mathcal{S}$ of strips which is admissible for (U, V).*

(D) *There exists a sequence $\{f_n\}_{n=1}^\infty$ of elements of $\mathcal{G}(X)$ such that*

  (1) $U \cap PC(\{f_n\}_{n=1}^\infty) = U$,
  (2) $U \cap CC(\{f_n\}_{n=1}^\infty) = V$,
  (3) $\lim_{n\to\infty} f_n = 0$ *on* U.

Proof: (A) $\Rightarrow$ (B) follows from Lemma 2.
(B) $\Rightarrow$ (C) follows from Theorem 9.
(C) $\Rightarrow$ (D) follows from Theorem 7.
(D) $\Rightarrow$ (A) follows from the fact that $\mathcal{G}$ is a sheaf. $\square$

Applications

For any $k \in \mathbb{N}$ the symbol $\mathcal{C}^k$ refers to a set of functions which are k-times continuously differentiable, and $\mathcal{C}^{k,1}$ refers to a set of k-times continuously differentiable functions with k-th partial derivatives locally Lipschitz.

Now we shall recall several terms. Let $d \geq 2$, $U \subset \mathbb{R}^d$ open. Let L be a differential operator on $\mathcal{C}^2(U)$ such that

$$Lu(x) := \sum_{j,k=1}^{d} a_{jk}(x) \cdot \frac{\partial^2 u(x)}{\partial x_j \partial x_k} + \sum_{i=1}^{d} b_i(x) \cdot \frac{\partial u(x)}{\partial x_i} + c(x) \cdot u(x),$$

where $x = (x_1, \dots, x_d)$ and coefficients are functions defined on U. We say that L is *elliptic* on U if $A(x) := \big(a_{jk}(x)\big)_{j,k=1}^{d}$ is a symmetric and strictly positive definite matrix for any $x \in U$. We say that L is *strictly elliptic* on U if A is a symmetric and locally uniformly strictly positive definite matrix on U. Any function $u \in \mathcal{C}^2(U)$ such that $Lu = 0$ on U is called *L-harmonic* on U.

Remark 11. *Using the theory of harmonic spaces* (*for details see* [CC]), *one can show that the following assertions hold:*

(a) *If L is an elliptic operator on $\mathbb{R}^d$ with locally Lipschitz coefficients, then $\mathbb{R}^d$ possesses the Vitali property with respect to the sheaf of L-harmonic functions.*
(b) *Let L be an elliptic operator on $\mathbb{R}^d$ such that $a_{jk} \in \mathcal{C}^{2,1}$, $b_i \in \mathcal{C}^{1,1}$, $c \in \mathcal{C}^{0,1}$ and $c \le 0$. Then the sheaf of L-harmonic functions is a Runge sheaf on $\mathbb{R}^d$.*
(c) *If L is an elliptic operator with continuous coefficients such that a_{jk} are Dini continuous and $c \le 0$, then $\mathbb{R}^d$ possesses the maximum property with respect to the sheaf of L-harmonic functions.*
(d) *If L is a strictly elliptic operator with locally bounded coefficients and $c \le 0$, then $\mathbb{R}^d$ possesses the maximum property with respect to the sheaf of L-harmonic functions.*

These assertions follow from [He, p. 560], [Pra, p. 398] *and* [CC, p. 79], *using* Remark 12(b)–(d), *and from* [BM, p. 40].

Remark 12. *The following holds:*

(a) *Let X be a harmonic space* (*in the sense of Constantinescu-Cornea*)*. By definition, X is locally compact and the corresponding sheaf of harmonic functions is non-degenerate and continuous. Moreover, X is locally connected.*
(b) *Let X be a harmonic space with countable base. Then X possesses the Vitali property with respect to the sheaf of harmonic functions.*
(c) *Let X be a connected harmonic space with countable base which has a base of completely determining domains. Suppose further that 1 is superharmonic and that there is a positive potential on any relatively compact domain in X. Assume that potentials with the same point support are proportional. Further we assume axiom A^* of quasi-analyticity. Then the sheaf of harmonic functions is a Runge sheaf on X.*
(d) *Let X be a Brelot harmonic space such that 1 is superharmonic. Then X possesses the maximum property with respect to the sheaf of harmonic functions.*
(e) *Let X be a harmonic space such that every point of X is polar. Then, for any domain Z of X and any $z \in Z$, $Z \setminus \{z\}$ is a domain.*

For (a) *see the definition of a harmonic space in* [CC, p. 30] *and* [CC, p. 11]. *Part* (b) *follows from* [CC, p. 272] *and Arzela-Ascoli theorem* [Con, p. 148]. *The proof of* (c) *can be found in* [GGG]. *For* (e), *see* [CC, p. 145]. *We shall prove* (d). *For any harmonic function g, $|g|$ is subharmonic. Since 1 is superharmonic, $|g| - \alpha$*

is subharmonic for any $\alpha > 0$. The maximum principle for subharmonic functions implies the maximum property (*see* [CC, p. 25]).

Theorem 13. *The sheaf of holomorphic functions on $\mathbb{C}$ is a non-degenerate continuous Runge sheaf on $\mathbb{C}$, and $\mathbb{C}$ possesses the maximum property and the Vitali property with respect to this sheaf.*
If L is an elliptic operator with continuous coefficients such that $a_{jk} \in \mathcal{C}^{2,1}$, $b_i \in \mathcal{C}^{1,1}$, $c \in \mathcal{C}^{0,1}$, a_{jk} Dini continuous and $c \leq 0$, then the sheaf of L-harmonic functions is a non-degenerate continuous Runge sheaf on $\mathbb{R}^d$, and $\mathbb{R}^d$ possesses the maximum property and the Vitali property with respect to this sheaf.
If L is a strictly elliptic operator such that $a_{jk} \in \mathcal{C}^{2,1}$, $b_i \in \mathcal{C}^{1,1}$, $c \in \mathcal{C}^{0,1}$ and $c \leq 0$, then the sheaf of L-harmonic functions is a non-degenerate continuous Runge sheaf on $\mathbb{R}^d$, and $\mathbb{R}^d$ possesses the maximum property and the Vitali property with respect to this sheaf.

Proof: On $\mathbb{C}$, the proof follows from Runge's theorem ([Con, p. 198]), Vitali's theorem ([Tit, p. 168]) and the maximum principle for holomorphic functions ([Con, p. 128]). On $\mathbb{R}^d$, the proof follows from Remark 11. $\square$

For the rest of this paper, L will be an elliptic or strictly elliptic operator satisfying the hypotheses of Theorem 13.

Theorem 14. *Suppose that $V \subset U$ are open subsets of $\mathbb{R}^2 \equiv \mathbb{C}$ such that no connected component of $\mathbb{R}^2 \setminus U$ is bounded. Then the following assertions are equivalent:*

(A$_1$) *There exists a sequence $\{f_n\}_{n=1}^{\infty}$ of holomorphic functions on U such that*

(1) $PC(\{f_n\}_{n=1}^{\infty}) = U$,
(2) $CC(\{f_n\}_{n=1}^{\infty}) = V$.

(A$_2$) *There exists a sequence $\{f_n\}_{n=1}^{\infty}$ of L-harmonic functions on U such that*

(1) $PC(\{f_n\}_{n=1}^{\infty}) = U$,
(2) $CC(\{f_n\}_{n=1}^{\infty}) = V$.

Proof: Suppose (A$_1$). It follows from Theorem 10 that (A$_1$) is equivalent to (C) for $X = \mathbb{C}$. The same argument proves the equivalence of (A$_2$) and (C) for $X = \mathbb{R}^2$. Since $\mathbb{R}^2 \equiv \mathbb{C}$, the proof is complete. $\square$

Theorem 15. *Let $V \subset U$ be open subsets of $\mathbb{C}$ such that no connected component of $\mathbb{C} \setminus U$ is bounded. If there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of holomorphic functions on U such that $PC(\{f_n\}_{n=1}^{\infty}) = U$ and $CC(\{f_n\}_{n=1}^{\infty}) = V$, then V is open and dense in U. Moreover, for any holomorphic function g on U there exists a sequence $\{g_n\}_{n=1}^{\infty}$ of holomorphic functions on U such that $g = \lim_{n\to\infty} g_n$ on U and $CC(\{g_n\}_{n=1}^{\infty}) = V$.*
Let $V \subset U$ be open subsets of $\mathbb{R}^d$ such that no connected component of $\mathbb{R}^d \setminus U$ is bounded. If there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of L-harmonic functions on U

such that $PC(\{f_n\}_{n=1}^{\infty}) = U$ *and* $CC(\{f_n\}_{n=1}^{\infty}) = V$*, then* V *is open and dense in* U*. Moreover, for any* L*-harmonic function* g *on* U *there exists a sequence* $\{g_n\}_{n=1}^{\infty}$ *of* L*-harmonic functions on* U *such that* $g = \lim_{n\to\infty} g_n$ *on* U *and* $CC(\{g_n\}_{n=1}^{\infty}) = V$*.*

Proof: It follows from Theorem 8 that V is open and dense in U. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence from Theorem 10, part (D). Define $\{g_n\}_{n=1}^{\infty} := \{g + f_n\}_{n=1}^{\infty}$. $\square$

Now we will present several criteria for sets U and V which will tell us when these sets allow the existence of a family admissible for (U, V).

Theorem 16. *Let* X *be a topological space such that there exists a sheaf* $\mathcal{G}$ *for which the pair* $(X, \mathcal{G})$ *satisfies hypotheses of* Theorem 10. *Let* $V \subset U$ *be open subsets of* X*. Define* $F := U \setminus V$*. Suppose that one of the following conditions holds.*

(a) V *is not dense in* U*.*
(b) *There exists a relatively compact domain* Z *in* X *such that* $\overline{Z} \subset U$*,* $F \cap Z \neq \emptyset$ *and* $F \cap \partial Z = \emptyset$*.*

Then no family of strips is admissible for (U, V)*.*

Proof: Part (a) follows from Theorem 8. Part (b) follows from the maximum property on Z and from Theorem 10, (C) $\Rightarrow$(A). $\square$

To simplify the notation in the next theorem, we shall introduce the following definitions: Suppose that $\psi : \langle 0, 1\rangle \to \mathbb{C} \cup \{\infty\}$ is a curve (i.e., a continuous mapping). For any $z \in \mathbb{C}$ and any $n \in \mathbb{N}$ define

$$P(\psi, z) := \{\ w + \alpha z\ ;\ w \in \psi(\langle 0, 1\rangle),\ \alpha \in (0, 1\rangle\ \},$$

$$P(\psi, z, n) := \{\ w + \alpha z\ ;\ w \in \psi(\langle 0, 1\rangle),\ \alpha \in \langle \frac{1}{3^n}, \frac{1}{2^n}\rangle\ \}.$$

Let γ, ϕ be curves on $\langle 0, 1\rangle$ such that $\gamma(1) = \phi(0)$. Then define

$$(\gamma\dot{+}\phi)(t) := \begin{cases} \gamma(2t) & \text{for } \ t \in \langle 0, \frac{1}{2}\rangle, \\[2ex] \phi(2t-1) & \text{for } \ t \in \langle \frac{1}{2}, 1\rangle. \end{cases}$$

Theorem 17. *Let* X *be a topological space such that there exists a sheaf* $\mathcal{G}$ *for which the pair* $(X, \mathcal{G})$ *satisfies the hypotheses of* Theorem 10. *Assume* $V \subset U$ *are open subsets of* X *such that no connected component of* $X \setminus U$ *is relatively compact. Define* $F := U \setminus V$*. Suppose that one of the following conditions holds.*

(a) *The set* F *has finitely many connected components* $F_1, \dots, F_k$*. For any* $j \le k$ *there exists a family of strips which is admissible for* $(U, U \setminus F_j)$*.*
(b) *Let* $X = \mathbb{C}$*,* $\mathbb{C} \setminus \overline{U}$ *connected. Suppose* $\gamma : \langle 0, 1\rangle \to \mathbb{C} \cup \{\infty\}$ *is an injective curve such that* F *is an intersection of* U *and* $\gamma\big(\langle 0, 1\rangle\big)$*,* $\gamma(1) \in \partial U$ *and*

$\gamma\big((0,1)\big) \subset U$. In case $\gamma(1) \neq \infty$ assume further that there exist $z \in \mathbb{C}$ and an injective curve $\phi : \langle 0,1\rangle \to \mathbb{C} \cup \{\infty\}$ such that $\phi(0) = \gamma(1)$, $\phi(1) = \infty$, $\infty \notin \phi((0,1))$, $\phi((0,1\rangle) \cap U = \emptyset$, and $P(\gamma \dot{+} \phi, z) \cap (\gamma \dot{+} \phi)(\langle 0,1\rangle) = \emptyset$.

Then there exists a family of strips which is admissible for (U,V).

Proof: First we prove (a). Using Theorem 10, for any $j \leq k$ we obtain a sequence $\{f_{j,n}\}_{n=1}^{\infty}$ of elements of $\mathcal{G}(U)$ such that $PC(\{f_{j,n}\}_{n=1}^{\infty}) = U$, $CC(\{f_{j,n}\}_{n=1}^{\infty}) = U \setminus F_j$. Define

$$f_n := \sum_{j=1}^{k} f_{j,n}$$

for any $n \in \mathbb{N}$. Then $PC(\{f_n\}_{n=1}^{\infty}) = U$, $CC(\{f_n\}_{n=1}^{\infty}) = V$ and Theorem 10 gives us a family of strips which is admissible for (U,V).

Now we prove (b). In case $\gamma(1) = \infty$ define $\phi := \infty$ on $\langle 0,1\rangle$. For any $n \in \mathbb{N}$ define

$$S_n := P(\gamma \dot{+} \phi, z, n) \cap \mathbb{C}.$$

Then $\mathcal{S} := \{S_n \; ; \; n \in \mathbb{N}\}$ is a family of strips which is admissible for (U,V). $\square$

Example

Let C be Cantor's ternary set on $\langle 0,1\rangle \subset \mathbb{R}$. If we define $G := \langle 0,1\rangle \setminus C$, then $G = \bigcup_{n=1}^{\infty} I_n$, where I_n are open intervals in $(0,1)$ and $I_k \cap I_j = \emptyset$ for $k \neq j$. Take $d \geq 2$ and define

$$S_n := \big(\, \overline{I_n} \times \langle 0,1\rangle^{d-2} \times \langle 0,\infty) \, \big) \; \cup \; \bigcup_{k=1}^{d-1} \big(\, \langle 0,1\rangle^{k} \times \overline{I_n} \times \langle 0,1\rangle^{d-1-k} \, \big).$$

Then $\mathcal{S} := \{S_n \; ; \; n \in \mathbb{N}\}$ is a family of strips which is admissible for $\big((0,1)^d, G^d\big)$.

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