Arithmetical classification of the set of all provably recursive functions

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Abstract. The set of all indices of all functions provably recursive in any reasonable theory T is shown to be recursively isomorphic to $U \times \overline{U}$, where U is Π_2-complete set.

Keywords: provable, recursive, complete
Classification: 03F30, 03D55

Let *arithmetical language* be the language $\{+, \cdot, 0, S, <\}$ with symbols for addition, multiplication, zero, successor and ordering, let *standard model* of arithmetic be the structure $\mathbf{N} = \langle N, +^{\mathbf{N}}, \cdot^{\mathbf{N}}, 0^{\mathbf{N}}, S^{\mathbf{N}}, <^{\mathbf{N}} \rangle$ of natural numbers. Let $\overline{n}$, the n-th *numeral*, be the closed term $S(S(..(0)..)$ with n occurrences of the symbol S. A set $A \subseteq N^k$ is *definable* in $\mathbf{N}$ if A has the form $\{ [n_1, .., n_k] ; \mathbf{N} \models \varphi(\overline{n_1}, .., \overline{n_k}) \}$ for some arithmetical formula $\varphi(x_1, .., x_k)$. A *classical result* (which can be seen as a version of Gödel First Incompleteness Theorem, see e.g. [7]) says that the recursively enumerable (r.e.) sets are exactly those subsets of N^k that are definable in $\mathbf{N}$ by Σ_1-formulas. Σ_1-*formulas* are formulas of the form $\exists v \lambda(x_1, .., x_k, v)$ where λ is bounded, and *bounded* formulas are formulas containing only quantifiers of the form $\forall x{<}y$ or $\exists x{<}y$ (i.e. containing only bounded quantifiers).

An axiomatic theory T *contains* Robinson's arithmetic Q if the language of T contains the arithmetical language and all axioms of Q are provable in T. A theory T is Σ_1-*sound* if all Σ_1-sentences provable in T hold in $\mathbf{N}$. For the rest of the paper a *theory* means a recursively axiomatizable Σ_1-sound theory containing Q.

Definition 1. *A function $f : N \to N$ is* provably recursive in T *if there exists a Σ_1-formula $\varphi(x, y)$ such that*

(i) $f = \{ [n, m] ; \mathbf{N} \models \varphi(\overline{n}, \overline{m}) \}$, *i.e. φ defines the graph of f in $\mathbf{N}$.*
(ii) $T \vdash \forall x \exists! y \varphi(x, y)$.

By the classical result mentioned above any function provably recursive in T has r.e. graph and hence is recursive. Thus functions provably recursive in T can be viewed as those recursive functions the totality of which is known to the theory T. It is worth mentioning that if φ and f are as in Definition 1 then the

This paper was supported by grants 162/97 of the Charles University and 401/98/0383 of the Grant Agency of the Czech Republic.

equivalence $f(n) = m \Leftrightarrow T \vdash \varphi(\overline{n}, \overline{m})$ holds for any pair n, m of natural numbers: $\Rightarrow$ is Σ-completeness, $\Leftarrow$ follows from Σ_1-soundness of T.

For T being Peano arithmetic PA, powerful methods capable of showing that some particular recursive functions are not provably recursive were developed in the late 70-ties ([8]) and later. Nice examples are e.g. in [5], more can be found in [4] or in [2]. Similar results were obtained also for subsystems of PA and for other theories. The importance of provably recursive functions lies in the fact that if f is recursive but not provably recursive in T then the statement the function f is total is an example of a true statement unprovable in T. Thus the methods for showing that some particular recursive function is not provably recursive in T are a source of independent statements, a source which is an alternative to the Gödel Second Incompleteness Theorem and which can yield statements that are more interesting from the "mathematical" point of view.

Can the existence of recursive functions that are not provably recursive in T be shown by a structural argument, i.e. by showing that the two sets

$$\{\, f\,;\, f \text{ is total} \,\} \quad \text{and} \quad \{\, f\,;\, f \text{ is provably recursive in } T \,\}$$

have index sets with different arithmetical classifications? We show that it indeed is the case. While the index set of the former set is known to be Π_2, we shall show that the index set of the latter set is neither Π_2 nor Σ_2.

In [6] the set $\Omega\mathit{Bound}$ of all indices of all general recursive functions with bounded range is investigated and its precise position in arithmetical hierarchy is found. The result obtained for $\Omega\mathit{Bound}$ is in [6] extended to some other index sets and could be extended also to the index set of all provably recursive functions. Thus our results cannot be claimed to be completely new. Rather, we present a logical version of a proof from [6] and apply it to index sets not mentioned in [6].

Lemma 1. *There exists a general recursive function $h : N^2 \to N$ which is universal for the set of all functions provably recursive in T. More specifically, the set $\{\, h(a, \cdot)\,;\, a \in N \,\}$ equals the set of all functions that are provably recursive in T.*

Here an in the sequel by $h(a, \cdot)$ we mean the function $n \mapsto h(a, n)$ (with a being constant). This function is sometimes also denoted by $\lambda n h(a, n)$. A simple diagonal argument shows that h is not provably recursive in T.

Proof of Lemma 1: Consider the following algorithm to compute h:

> Read inputs a and n.
> Find least $d \geq a$ such that d is a proof in T of some sentence of the form $\forall x \, \exists!\, y \varphi(x, y)$ with φ in Σ_1.
> Find m such that $\mathbf{N} \models \varphi(\overline{n}, \overline{m})$. Output the number m.

It is easy to verify that h has the desired properties. □

Let, as in [9], φ_e be the e-th partial recursive function and W_e be the e-th r.e. set. Let U be the set $\{\, e\,;\, W_e \text{ is infinite} \,\}$. The set U is known to be Π_2-complete. For $A, B \subseteq \mathbf{N}$, let $A \times B$ denote the set $\{\, c(i, j)\,;\, i \in A \text{ and } j \in B \,\}$ where c is the pairing function, and let $\overline{A}$ denote the complement of A.

Theorem 1. *The set $A = \{ e ; \varphi_e$ is provably recursive in $T \}$ is recursively isomorphic to $U \times \overline{U}$.*

Proof: Since A is a cylinder (verification is left to the reader) it is sufficient to prove $A \leq_{\mathrm{m}} U \times \overline{U}$ and $U \times \overline{U} \leq_{\mathrm{m}} A$.

Let h be the function from Lemma 1. For any e, the function φ_e is provably recursive if and only if

$$\varphi_e \text{ is total } \ \& \ \exists a \forall x (\varphi_e(x) = h(a, x)),$$

where the left conjunct is known to be Π_2 and the right one is Σ_2 (the condition $\varphi_e(x) = h(a, x)$ is Π_1 because it says that any computation of φ_e on input x yields the result $h(a, x)$). Thus A is an intersection of a Π_2- and a Σ_2-set. Since U is Π_2-complete and $\overline{U}$ is Σ_2-complete it is evident that $A \leq_{\mathrm{m}} U \times \overline{U}$.

To prove $U \times \overline{U} \leq_{\mathrm{m}} A$ it is sufficient to find a partial recursive function ψ of three variables such that, for each x and y, the function $\psi(x, y, \cdot)$ is provably recursive in T iff W_x is infinite and W_y is finite. Consider the following algorithm to compute ψ:

Read inputs x, y and v.
Find an element of W_x which is greater than v.
Find $a := \sup\{ z ; \exists w \leq v T(y, z, w) \}$.
Output the number $1 + \max\{h(0, v), .., h(a, v)\}$.

Here $T(y, z, w)$ is the Turing predicate. T is primitive recursive and satisfies $W_y = \{ z ; \exists w T(y, z, w) \}$ for each y. We suppose that for each x and w there is at most one z such that $T(y, z, w)$. Hence the set in the third line of our algorithm is finite and the instruction "Find $a := \sup\{..\}$" is correct. Note that the algorithm does nothing with the element of W_x found in the second line. This instruction is there only to ensure that the algorithm starts cycling in cases it is supposed to do so. The function ψ can be verified to have the following properties:

- the function $\psi(x, y, \cdot)$ is total iff W_x is infinite
- if W_x is infinite, W_y is finite and $a = \max W_y$ then $\psi(x, y, \cdot)$ differs from the function $v \mapsto 1 + \max\{h(0, v), .., h(a, v)\}$ on a finite set and hence is provably recursive
- if W_x and W_y are both infinite then $\psi(x, y, \cdot)$ is total but different from all functions $h(a, \cdot)$, $a \in N$.

Thus ψ is as required. □

Besides the fact that recursive functions that are not provably recursive in T do exist for each theory T in question (which follows already from Lemma 1) we mention two further consequences of our theorem.

Corollary 1. *It is known, see e.g.* [4] *or* [2], *that primitive recursive functions are exactly those functions that are provably recursive in $I\Sigma_1$, where $I\Sigma_1$ is Peano arithmetic with the induction scheme restricted to Σ_1-formulas. Thus it follows* *from Theorem 1 that the set of all indices of all primitive recursive functions is recursively isomorphic to* $U \times \overline{U}$.

Corollary 2. *S. Buss proved in* [1] *that the polynomial time computable functions are exactly those functions that are* Σ_1^{b}*-definable in the theory* S_2^1. *A function* f *is* Σ_1^{b}*-definable in* T *if there is a* Σ_1^{b}*-definition* $\varphi(x,y)$ *of its graph such that* $T \vdash \forall x \, \exists! \, y \varphi(x,y)$ *(see* [1] *for the definition of the theory* S_2^1*, for the definition of* Σ_1^{b}*-formulas and for more information). An inspection of our proof shows that it works also for* Σ_1^{b}*-definable functions. Thus we have two sets of functions connected to the theory* S_2^1*: all polynomial time computable (i.e.* Σ_1^{b}*-definable in* S_2^1*) functions, and all functions provably recursive in* S_2^1*. We cannot claim that these two sets are equal, but each has an index set recursively isomorphic to* $U \times \overline{U}$.

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(*Received* February 4, 1999, *revised* April 12, 1999)