# Solutions to a perturbed critical semilinear equation concerning the $N$-Laplacian in $\mathbb{R}^N$

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*Abstract.* The aim of this paper is to study the existence of variational solutions to a nonhomogeneous elliptic equation involving the $N$-Laplacian

$$-\Delta_N u \equiv -\operatorname{div}(|\nabla u|^{N-2}\nabla u) = e(x,u) + h(x) \text{ in } \Omega$$

where $u \in W_0^{1,N}(\mathbb{R}^N)$, $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$, $e(x,u)$ is a critical nonlinearity in the sense of the Trudinger-Moser inequality and $h(x) \in (W_0^{1,N})^*$ is a small perturbation.

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## 1. Introduction

Let $\Omega$ be a smooth bounded set in $\mathbb{R}^N$, $N \geq 2$, and consider the problem

$$\begin{aligned} -\Delta_N u &= e(x,u) + h(x) \\ u &\in W_0^{1,N}(\Omega) \end{aligned} \tag{1}$$

where $e(x,u)$ is a critical function in terms of the Trudinger-Moser inequality and $h \in W^{-1,N'}$. Such a nonlinearity $e(x,u)$ possesses the maximal growth in $u$ which permits a variational formulation of problem (1).

Solutions are sought in the Sobolev space $W_0^{1,N}(\Omega)$, defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\| \equiv (\int_\Omega |\nabla u|^N)^{\frac{1}{N}}$. The dual space is denoted $W^{-1,N'}$, where $N'$ is the Hölder conjugate of $N$, and has the associated norm $\|\cdot\|_*$. Denote strong convergence by "$\to$", weak convergence by "$\rightharpoonup$" and convergence in the sense of measure (or distributions) as "$\rightharpoonup^*$". Unless otherwise denoted, integration is performed over the domain $\Omega$. Specific constraints on $e(x,u)$ and $h(x)$ are described later, but we now present the main results:

**Theorem 1.1.** *Suppose $E(x,u)$ is a function of critical growth satisfying* (7) *to* (11).

(i) *There exists $h^* > 0$ such that for each $h(x)$ with $0 < \|h\|_* < h^*$, problem* (1) *possesses a solution at negative energy.*

(ii) *If* $e(x,u)$ *further satisfies* (12) *then there exists a number* $h^{**} > 0$, *possibly smaller than* $h^*$ *from* (i), *such that for each* $h(x)$ *with* $0 < \|h\|_* < h^{**}$, *there exists another solution to* (1).

**Theorem 1.2.** *If the conditions of Theorem* 1.1 *hold and* $h(x) \geq 0$ $(h(x) \leq 0)$ *almost everywhere, then the solutions in* (i) *and* (ii) *are nonnegative* (*nonpositive*).

Weak solutions of (1) correspond to critical points of the functional $I$:

$$I(u) = \frac{1}{N}\int_\Omega |\nabla u|^N \, dx - \int_\Omega E(x,u)\, dx - \int_\Omega hu\, dx \tag{2}$$

where $E(x,u) = \int_0^u e(x,t)\, dt$. At this stage we introduce an associated functional

$$I^+(u) = \frac{1}{N}\int |\nabla u|^N - \int E^+(x,u) - \int hu$$

where $E^+(x,u)$ corresponds with $E(x,u)$ when $u \geq 0$, but is otherwise set to zero. Critical points of $I^+$ correspond to solutions in $W_0^{1,N}(\Omega)$ of

$$-\Delta_N u = e^+(x,u) + h(x). \tag{3}$$

It can be shown ([14]) that both $I^+(u)$ and $I(u) \in C^1(W_0^{1,N};\mathbb{R})$.

Publication [13] has considered problem (1) with $h(x) \equiv 0$. Much of the geometrical structure captured in this analysis still holds, and this paper includes useful convergence lemmas. The geometry of the functional allows application of the Mountain-Pass theorem of Ambrosetti-Rabinowitz, without the Palais-Smale condition.

In [13], to prove that Palais-Smale sequences expose solutions, a weakly convergent sequence is shown to converge to a nontrivial solution. This method elicits no further information.

In this paper, analogous arguments may be made. For the unperturbed problem, $u = 0$ is a local minimum. For small $\|h\|_*$, we anticipate a local minimum solution near zero, and this is located via a local minimisation technique. A perturbed solution close to the non-trivial solution derived in [13] is also expected. We derive a solution from a mountain pass technique, but the lack of a Palais-Smale condition means that strong convergence is not assured. Indeed, the lack of a (PS) condition resulting from a critical nonlinearity makes it difficult to prove that these two solutions are not identical. To distinguish two solutions, a distinction result is achieved based on the difference in sequence energies and P.L. Lions' theorem.

In a similarly perturbed problem, Deng and Li [7] show the existence of solutions without a Palais-Smale condition, and a distinction between solutions, but do not go so far as to allege strong convergence. The maximum principle is used to show positivity of solutions, but this technique fails for the $N$-Laplacian case.

The Trudinger-Moser [17], [12] inequality says that

$$\exp\left(\alpha|u|^{\frac{N}{N-1}}\right) \in L^1(\Omega)\ \forall\, u \in W_0^{1,N}(\Omega),\ \forall\, \alpha > 0 \tag{4}$$

$$\sup_{\|u\|\le 1} \int_\Omega \exp\left(\alpha|u|^{\frac{N}{N-1}}\right) \le C(N) \in \mathbb{R} \ \text{ if } \ \alpha \le \alpha_N \tag{5}$$

where $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$ and $\omega_{N-1}$ is the volume of the $(N-1)$ dimensional surface of the unit sphere and $C(N)$ is a constant depending only on $N$.

This result is stronger than the Sobolev embedding, which expresses that $W_0^{1,N}(\Omega) \hookrightarrow L^t(\Omega)$ compactly for all $t \ge 1$, but *not* $L^\infty(\Omega)$.

Carleson and Chang [5] have shown that when $\Omega$ is a ball, the extremal function for this inequality is achieved in $W_0^{1,N}(\Omega)$. Recently Lin [10], has extended this result to general domains $\Omega \subset \mathbb{R}^N$. This contrasts with the case of critical functions for the embeddings $W_0^{1,2}$ into the space $L^{\frac{2N}{N-2}}$. The so-called Talenti extremal functions are scale and translation invariant and rely on an unbounded domain.

The Trudinger-Moser inequalities can be improved in a theorem by Lions [11]:

**Theorem 1.3.** *Let $\{u_n : \|u_n\| = 1\}$ be a sequence in $W_0^{1,N}$ converging weakly to a nonzero function $u$. Then, for every $p < (1 - \|u\|^N)^{\frac{-1}{N-1}}$ we have*

$$\sup_n \int_\Omega \exp\left(p\alpha_N|u_n|^{\frac{N}{N-1}}\right)\, dx < \infty. \tag{6}$$

Theorem 1.3 improves the Trudinger-Moser inequality (5) by accounting for the possibility of concentration. Suppose $\{u_n\} \subset W_0^{1,N}$, $\|u_n\| = 1$, $u_n \rightharpoonup u_0 \ne 0$. If $u_n \not\to u_0$ strongly, then $\|u_n\|^N = \|u_0\|^N + \|v_n\|^N + o(1)$ where $v_n \rightharpoonup 0$ but $\lim_{n\to\infty} \|v_n\| > 0$ contains the concentrations. Consequently, $\|u_0\| < 1$ and $(1 - \|u_0\|^N)^{\frac{-1}{N-1}} > 1$. Thus expression (6) improves (5) by allowing a larger exponent. If $u_n \rightharpoonup 0$ then the two results correspond. If $u_n \to u_0$ then $(1 - \|u_0\|^N)^{\frac{-1}{N-1}} = \infty$ and $\lim_{n\to\infty} \int \exp(p\alpha_N|u_n|^{\frac{N}{N-1}}) < \infty$ for any $p > 0$.

### 1.1 Assumptions

The assumptions on the nonlinearity $e(x, u)$ will be altered slightly from the version in [13] to accommodate negative solutions. Essentially we impose symmetric constraints on $e(x, u)$. Of course, these can be lifted if we neglect interest in signs of solutions, and a remark to this effect is made later.

Make the following assumptions on $e(x, u)$:

Assume $e(x,u)$ is a critical function with exponent $\alpha_0$, so that

$$\begin{aligned}
&\lim_{|u|\to\infty} \frac{e(x,u)}{\exp(\alpha|u|^{\frac{N}{N-1}})} = 0 \quad \text{for} \quad \alpha > \alpha_0; \\
(7) \qquad &\lim_{|u|\to\infty} \frac{|e(x,u)|}{\exp(\alpha|u|^{\frac{N}{N-1}})} = \infty \quad \text{for} \quad \alpha < \alpha_0.
\end{aligned}$$

Assume the continuity and sign restrictions:

$$e(x,u) \in C(\Omega \times \mathbb{R}; \mathbb{R}); \tag{8}$$

$$e(x,u) \geq 0 \quad \text{on} \quad \Omega \times [0,\infty), \; e(x,u) \leq 0 \quad \text{on} \quad \Omega \times (-\infty, 0]. \tag{9}$$

Assume there exists $R > 0$ and $M > 0$ such that for all $|u| \geq R$ and $x \in \Omega$

$$0 < E(x,u) \leq M|e(x,u)|. \tag{10}$$

Further, make the assumption on $E(x,u)$ that

$$\limsup_{u\to 0} \frac{NE(x,u)}{|u|^N} < \lambda_1 \tag{11}$$

where $\lambda_1$ is the first eigenvalue of $-\Delta_N u = \lambda|u|^{N-2}u$ characterised by

$$\lambda_1 = \inf\left\{\int |\nabla u|^N : u \in W_0^{1,N}, \int |u|^N = 1\right\}.$$

As per [13], define

$$\mathcal{M} = \lim_{n\to\infty} n \int_0^1 \exp\left[n(t^{\frac{N}{N-1}} - t)\right] \, dt \geq 2.$$

Denote by $d$ the inner radius of $\Omega$. Introduce the condition that uniformly on $\Omega$,

$$\lim_{u\to\infty} ue(x,u) \exp\left(-\alpha_0|u|^{\frac{N}{N-1}}\right) \geq \beta_0 > \left(\frac{N}{d}\right)^N \frac{1}{\mathcal{M}\alpha_0^{N-1}}. \tag{12}$$

Another condition which we shall find useful is that uniformly on $\Omega$,

$$\lim_{u\to\pm\infty} ue(x,u) \exp\left(-\alpha_0|u|^{\frac{N}{N-1}}\right) \geq \beta_0 > \left(\frac{N}{d}\right)^N \frac{1}{\mathcal{M}\alpha_0^{N-1}}. \tag{13}$$

In publications such as [1], [2] and [15] restrictions imposed on $e(x,u)$ are of the form

$$\frac{\partial e(x,t)}{\partial t} > \frac{e(x,t)}{t}.$$

Throughout this work, we discard this restriction in favour of the restraints posed in [13]. The definition of a critical function $e(x,u)$ (in (7)) compares $e(x,u)$ with $\exp(\alpha|u|^{\frac{N}{N-1}})$ at infinity when $\alpha < \alpha_0$ and $\alpha > \alpha_0$. Condition (12) fills in the gap by comparing $e(x,u)$ with $\exp(\alpha_0|u|^{\frac{N}{N-1}})$.

### 1.2 Direct results from assumptions

For a critical function $e(x,u)$, for any $\beta > \alpha_0$, there exists $C > 0$ such that

$$|e(x,u)| \le C \exp\left(\beta |u|^{\frac{N}{N-1}}\right).$$

There is a $C > 0$ such that for $|u| \ge R$, and all $x \in \Omega$

$$E(x,u) \ge C \exp\left(\frac{1}{M} u\right). \tag{14}$$

There is $R_0 > 0$ and $\theta > N$ such that for $|u| \ge R_0$ and $x \in \Omega$,

$$\theta E(x,u) \le u e(x,u). \tag{15}$$

From these, we can deduce that for fixed $q > N$, fixed $\lambda < \lambda_1(N)$ and fixed $\beta > \alpha_0$, there is some $C > 0$ such that

$$E(x,u) \le \frac{1}{N}\lambda |u|^N + C|u|^q \exp\left(\beta |u|^{\frac{N}{N-1}}\right). \tag{16}$$

## 2. Geometry of the functional

Throughout, we assume that the $E(x,u)$ satisfies (7) to (11). From line to line, constants are denoted $C$ but may assume different values. This section has the two-fold aim of analysing the geometry of $I$ and $I^+$.

**Lemma 2.1.** (i) *There exists a number $h^* > 0$ such that for each $h(x)$ with $\|h\|_* < h^*$, there exists $\rho_h > 0$ such that the functional $I$ satisfies*

$$I(u) > 0 \quad \forall\, u \in W_0^{1,N}, \ \|u\| = \rho_h.$$

*Furthermore, $\rho_h$ may be chosen such that $\rho_h \to 0$ as $\|h\|_* \to 0$.*

(ii) *The same result holds for $I^+(u)$.*

Proof: (i) To develop the mountain ridge we estimate $I(u)$ from below on a $\rho$-ball in $W_0^{1,N}$. Fix $\lambda < \lambda_1$ and $\beta > \alpha_0$. By implementing estimate (16),

$$\begin{aligned} I(u) &\ge \frac{1}{N}\int |\nabla u|^N - \frac{\lambda}{N}\int |u|^N - C\int \exp\left(\beta |u|^{\frac{N}{N-1}}\right)|u|^q - \|h\|_*\|u\| \\ &\ge \frac{1}{N}\left(1 - \frac{\lambda}{\lambda_1}\right)\int |\nabla u|^N - C\int \exp\left(\beta |u|^{\frac{N}{N-1}}\right)|u|^q - \|h\|_*\|u\|. \end{aligned}$$

Choose any $r > 1$. Then provided that

$$\|u\| < \left(\frac{\alpha_N}{\beta r}\right)^{\frac{N-1}{N}} \tag{17}$$

it follows from the Trudinger-Moser inequality that for $r^{-1} + s^{-1} = 1$,

$$\int \exp\left(\beta|u|^{\frac{N}{N-1}}\right)|u|^q \le \left[\int \exp\left(\beta r\|u\|^{\frac{N}{N-1}}\left|\frac{u}{\|u\|}\right|^{\frac{N}{N-1}}\right)\right]^{\frac{1}{r}}\left(\int |u|^{sq}\right)^{\frac{1}{s}}$$
$$\le C\left(\int |u|^{sq}\right)^{\frac{1}{s}}.$$

Hence,

$$I(u) \ge \frac{1}{N}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^N - C\|u\|_{sq}^q - \|h\|_*\|u\|.$$

Use the Sobolev embedding $W_0^{1,N} \hookrightarrow L^t\ \forall\, t \ge 1$, so $C\|u\| \ge \|u\|_{sq}$, to reveal

$$I(u) \ge \|u\|\left[\frac{1}{N}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^{N-1} - C\|u\|^{q-1} - \|h\|_*\right].$$

With $\|u\| = \rho$, the functional is estimated from below on a ball

$$I(u) \ge \rho\left[\frac{1}{N}\left(1 - \frac{\lambda}{\lambda_1}\right)\rho^{N-1} - C\rho^{q-1} - \|h\|_*\right]. \tag{18}$$

Since $q > N$, it follows that if $\|h\|_*$ is sufficiently small then there exists some $\rho_h > 0$ such that $I(u) > 0$ on $B(0, \rho_h) \subset W_0^{1,N}$. As $\|h\|_*$ becomes smaller, expression (18) permits $\rho_h$ to be chosen commensurately smaller. In particular, $\rho_h$ is sufficiently small to ensure that restriction (17) is fulfilled.

(ii) The proof in the case of $I^+$ is identical. □

**Lemma 2.2.** (i) *There exists $\eta > 0$ sufficiently small such that*

$$\inf_{\|u\| \le \eta} I(u) < 0.$$

*Furthermore, there exists $\eta > 0$ and $u \in W_0^{1,N}$ with $\|u\| = 1$ such that $I(tu) < 0$ for all $0 < t < \eta$.*

(ii) *The same result holds for $I^+(u)$.*

Proof: (i) Choose $\tilde{u} \in W_0^{1,2}(\mathbb{R}^N)$ with $\|\tilde{u}\|_{W_0^{1,N}} = 1$ and $\int h\tilde{u} > 0$. For $t > 0$,

$$\frac{d}{dt}I(t\tilde{u}) = t^{N-1}\int |\nabla\tilde{u}|^N - \int e(x, t\tilde{u})\tilde{u} - \int h\tilde{u}.$$

As $e(x, \cdot)$ is continuous with $e(x, 0) = 0$, it follows that there exists $\eta > 0$ such that $\frac{d}{dt}I(t\tilde{u}) < 0$ for all $t < \eta$. Since $I(0) = 0$, it must hold that $I(t\tilde{u}) < 0$ for all $0 < t < \eta$.

(ii) The method of proof for $I^+$ is identical. □

**Lemma 2.3.** (i) *There exists* $u_b \in W_0^{1,N}$ *with* $\|u_b\| > \rho_h$ *and*

$$I(u_b) < \inf_{\|u\|=\rho_h} I(u).$$

(ii) *The same result holds for* $I^+$.

Proof: (i) From (14), for $p > N$, there are positive constants $C$ and $d$ such that for all $u \geq 0$,

$$E(x,u) \geq Cu^p - d.$$

For any $u \in W_0^{1,N} \setminus \{0\}$, have

$$I(tu) \leq \frac{1}{N} t^N \int |\nabla u|^N - Ct^p \int |u|^p + t\|h\|_* \|u\| + d.$$

As $t \to \infty$, $I(tu) \to -\infty$ and the result follows.

(ii) For a nonlinearity $E^+(x,u)$, equation (14) becomes

$$E^+(x,u) \geq C \exp\left(\frac{1}{M} u\right)$$

for all $u \geq R$ and $x \in \Omega$. The method of proof for $I^+$ then follows in an identical manner. □

We define the sequence of Moser [12] functions $M_n(x, x_0, r)$ in $W_0^{1,2}(\Omega)$. This often utilised family provides a large value of $\int \exp(\alpha_N |M_n|^{\frac{N}{N-1}})$, while maintaining $\|M_n\| = 1$. In many applications, this particular sequences poses problems as it is weakly convergent to zero. In fact [10], $M_n$ converges in the sense of distributions to a Dirac delta-function.

$$M_n(x, x_0, r) = \omega_{N-1}^{-\frac{1}{N}} \begin{cases} (\log n)^{\frac{N-1}{N}} & \text{if } 0 \leq |x - x_0| \leq \frac{r}{n} \\ \dfrac{\log\left|\frac{x-x_0}{r}\right|^{-1}}{(\log n)^{\frac{1}{N}}} & \text{if } \frac{r}{n} \leq |x - x_0| \leq r \\ 0 & \text{if } |x - x_0| \geq r. \end{cases}$$

For completeness, we reproduce Lemma 3 from [13].

**Lemma 2.4.** *Assume that* $e(x,u)$ *satisfies* (12). *Then there exists* $n \in \mathbb{N}$ *such that*

$$\max_{t \geq 0} \frac{t^N}{N} \int |\nabla M_n|^N - \int E(x, tM_n) < \frac{1}{N} \left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

**Remark 2.5.** *By construction,* $M_n(x) \geq 0$ *on* $\Omega$ *and so* $E(x, tM_n(x)) \equiv E^+(x, tM_n(x))$ *on* $\Omega$. *Lemma* 2.4 *holds for* $E$ *replaced with* $E^+$.

The following is a simple consequence of taking the negative half of $e(x,u)$, the negative limit in (13) and implementing Lemma 2.4.

**Corollary 2.6.** *Suppose $e(x,u)$ satisfies* (13)*. Then there exists $n \in \mathbb{N}$ such that*

$$\max_{t\geq 0} \frac{t^N}{N} \int |\nabla M_n|^N - \int E(x,-tM_n) < \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

These results still hold with a perturbation. Using the known results from Lemma 2.4 and Corollary 2.6 upper bounds on energy levels for the functional $I$ follow easily. This proof relies upon the limit in (13) tending to positive *or* negative infinity.

**Lemma 2.7.** (i) *If $e(x,u)$ satisfies* (12) *and $h(x) \geq 0$ almost everywhere then there exists $\tilde{u}(x) \in W_0^{1,N}$ such that*

$$I^+(t\tilde{u}), I(t\tilde{u}) < \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}$$

*for all $t \geq 0$.*

(ii) *If $e(x,u)$ satisfies* (13) *and $h(x)$ is of any sign then*

$$I(t\tilde{u}) < \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}$$

*for all $t \geq 0$.*

Proof: (i) Let $n > 0$ be the integer from Lemma 2.4. Since $\int hM_n > 0$ then there is nothing further to prove as $E(x,tM_n) = E^+(x,tM_n)$ and

$$\begin{aligned} I^+(tM_n) = I(tM_n) &= \frac{t^N}{N}\|M_n\| - \int E(x,tM_n) - t\int hM_n \\ &\leq \frac{t^N}{N}\|M_n\| - \int E(x,tM_n) < \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \end{aligned}$$

(ii) Suppose that $\int hM_m < 0$ for all $m > n$. Consider the sequence

$$\frac{t^N}{N}\int |\nabla M_m|^N - \int E(x,-tM_m).$$

Using Corollary 2.6, for sufficiently large $m \in \mathbb{N}$,

$$\begin{aligned} \max_{t\geq 0} I(-tM_m) &= \max_{t\geq 0}\left\{\frac{t^N}{N}\|tM_m\| - \int E(x,-tM_m) + t\int hM_m\right\} \\ &\leq \max_{t\geq 0}\left\{\frac{t^N}{N}\|tM_m\| - \int E(x,-tM_m)\right\} \\ &< \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \end{aligned}$$

□

**Remark 2.8.** *It appears possible to surmount technical difficulties and marginally improve on Lemma 2.7 by demanding only that $e(x,u)$ satisfy (12) irrespective of the sign of $h(x)$. The key to this proposal is to use the fact that $M_n \rightharpoonup 0$ in $W_0^{1,N}$, and hence the perturbation term $\int hM_n$ becomes arbitrarily small with large $n$.*

By the continuity of $I(u)$, it follows from Lemmas 2.1 and 2.2 that

$$-\infty < c_0 \equiv \inf\{I(u) : u \in W^{1,N}, \|u\| \le \rho_h\} < 0. \tag{19}$$

Later we prove that this infimum is achieved and elicits a solution.

In order to invoke the forthcoming convergence results, we require a tighter bound on the maximal energy of the functional than the expressions in Lemma 2.7. To control the magnitude of $I(u)$ along a mountain-pass path, the size of $\|h\|_*$ is restricted.

**Lemma 2.9.** (i) *Assume that $e(x,u)$ satisfies* (12) *and $h(x) \ge 0$; or*
(ii) *that $h(x)$ is indefinite in sign and $e(x,u)$ satisfies* (13).
*There exists $h^{**} > 0$ such that for all $0 \le h(x) \in W^{-1,N'}$ with $0 < \|h\|_* < h^{**}$ there is some $\tilde{u}(x) \in W_0^{1,N}$ with the property that for* (i)

$$I^+(t\tilde{u}), I(t\tilde{u}) < c_0 + \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}$$

*for all $t \ge 0$, while for* (ii),

$$I(t\tilde{u}) < c_0 + \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Proof: When $h \equiv 0$, Lemma 2.1 shows that the origin forms a local minimum of the functional $I$. Let $\rho > 0$ be chosen such that $J(u) = \frac{1}{N}\|u\| - \int E(x,u) > \delta$ for all $\|u\| = \rho$. Perturbing the functional $J(u)$ by the term $-\int hu$, we see that $I(u)$ will remain positive for all $\|u\| = \rho$ if $\|h\|_* < \frac{\delta}{\rho}$.

It is possible to raise the lower bound for $c_0$ by reducing $\|h\|_*$. By Lemma 2.1, $\rho_h \to 0$ as $\|h\|_* \to 0$. Consequently, the infimum of $I(u)$ on $B(0,\rho_h)$ is increasing and $c_0 \to 0$ as $\|h\|_* \to 0$.

For part (i) use Lemma 2.7(i) and for part (ii) use Lemma 2.7(ii) to see that $I^+(t\tilde{u}), I(t\tilde{u}) < [\frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1} - \epsilon]$ for some $\epsilon > 0$. Defining $h^{**}$ sufficiently small enforces $c_0 > -\epsilon$ and the results follow. $\square$

## 3. Convergence properties of sequences

The development of the Palais-Smale levels leads to the construction of a sequence which is not necessarily strongly convergent in $W_0^{1,N}$. Instead, weak convergence to a nontrivial weak solution is achieved for certain energies.

The following lemma is attributed to de Figueiredo *et al* [8]:

**Lemma 3.1.** *Let $\{u_n\} \subset L^1(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$ and $e(x,u)$ be continuous. Then $e(x,u_n) \to e(x,u)$ in $L^1(\Omega)$ provided that $e(x,u_n) \in L^1(\Omega)$ for all $n$ and $\int |e(x,u_n(x))u_n(x)| \le C_1$.*

**Lemma 3.2.** *For a $(PS)_c$ sequence $\{u_n\}$ for $I$ or $I^+$ at any level $c$, there is a subsequence relabelled $\{u_n\}$ and $u \in W_0^{1,N}$ such that*

$$e(x,u_n) \to e(x,u) \quad \textit{in} \quad L^1(\Omega); \tag{20}$$

$$E(x,u_n) \to E(x,u) \quad \textit{in} \quad L^1(\Omega); \tag{21}$$

$$|\nabla u_n|^{N-2}\nabla u_n \rightharpoonup |\nabla u|^{N-2}\nabla u \quad \textit{weakly in} \quad (L^{\frac{N}{N-1}}(\Omega))^N. \tag{22}$$

Proof: This entire proof works in an identical manner for $I$ and $I^+$. The following details concern $I$, but replacing $E$ and $e$ with $E^+$ and $e^+$ provides the analogous result.

Suppose $\{u_n\}$ is a Palais-Smale sequence at level $c$, so

$$\frac{1}{N}\int |\nabla u_n|^N - \int E(x,u_n) - \int h u_n \to c \tag{23}$$

$$\int |\nabla u_n|^{N-2}\nabla u_n.\nabla v - \int e(x,u_n)v - \int hv \to 0 \;\forall\, v \in W_0^{1,N}. \tag{24}$$

*Step 1*: Show $\{u_n\}$ bounded in $W_0^{1,N}$.

From (23) and (24) have that

$$\left|\left(\frac{\theta}{N}-1\right)\|u_n\|^N - \int [\theta E(x,u_n) - u_n e(x,u_n)] - (\theta-1)\int h u_n\right| \le C + \epsilon_n\|u_n\|$$

where $\epsilon_n \to 0$ as $n \to \infty$. Thus,

$$\left|\left[\left(\frac{\theta}{N}-1\right)\|u_n\|^{N-1} - (\theta-1)\,\|h\|_*\right]\|u_n\| - \int [\theta E(x,u_n) - u_n e(x,u_n)]\right| \le C + \epsilon_n\|u_n\|. \tag{25}$$

Using (25) with (15), we have that $\{u_n\}$ is bounded in $W_0^{1,N}(\Omega)$. From this,

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in} \quad W_0^{1,N} \\ u_n &\to u \quad \text{in} \quad L^q \;\forall\, q \ge 1 \\ u_n(x) &\to u(x) \quad \text{a.e. in } \Omega. \end{aligned}$$

*Step 2*: Claim $\{u_n\}$ has a subsequence such that (20) holds.

Justification of (20) follows from Lemma 3.1. To see the applicability of this lemma, note that $\|u_n\| \leq K$, so

$$-K\|h\|_* \leq \int hu_n \leq K\|h\|_*$$

then applying this to (23) and (24),

$$\int E(x,u_n) \leq C; \quad \text{and} \quad \left|\int e(x,u_n)u_n\right| \leq C. \tag{26}$$

Further, $e(x,u_n) \in L^1(\Omega)$ for all $n$ by the Trudinger-Moser inequality:

$$|e(x,u_n)| \leq C\exp(\beta|u_n|^{\frac{N}{N-1}}) \in L^1(\Omega) \ \text{ for each } \ u_n \in W_0^{1,N}.$$

*Step 3*: Claim (21) holds.

By condition (10), there exists $\overline{E} > 0$ such that

$$E(x,u) \leq \overline{E} + Me(x,u).$$

Now,

$$\int E(x,u_n)u_n \leq \int \overline{E}u_n + M\int e(x,u_n)u_n.$$

The first term is bounded as $u_n \to u$ in $L^1(\Omega)$, and the second term is bounded by (26). Consequently, the requirements are satisfied to invoke Lemma 3.1 on $E(x,u)$ to prove (21).

*Step 4*: Claim $\{u_n\}$ has a subsequence such that (22) holds.

Note that $u_n$ satisfies convergence weakly to a measure:

$$|\nabla u_n|^N \rightharpoonup^* \mu \ \text{ in } \ \mathcal{D}(\Omega)$$

$$|\nabla u_n|^{N-2}\nabla u_n \rightharpoonup V \ \text{ weakly in } \ (L^{\frac{N}{N-1}})^N$$

where $\mu$ is a regular finite measure and $\mathcal{D}(\Omega)$ are the distributions on $\Omega$.

Clearly $A_\sigma = \{x \in \overline{\Omega} : \mu(B_r(x) \cap \overline{\Omega}) \geq \sigma\}$ is a finite set, say $\{x_1, x_2, \ldots, x_m\}$, for otherwise $\mu(A_\sigma) = \infty$ contradicting $\mu(A_\sigma) = \lim_{k\to\infty} \int_{A_\sigma} |\nabla u_n|^N \leq C$.

**Assertion 1.** If we choose $\sigma > 0$ such that $\sigma^{\frac{1}{N-1}}\beta < \alpha_N$, then $e(x,u_n)u_n \to e(x,u)u$ in $L^1(K)$ where $K \subset \overline{\Omega} \cap A_\sigma$ is compact.

To prove the assertion, let $x_0 \in K$ and $r_0 > 0$ be such that $\mu(B_{r_0} \cap \overline{\Omega}) < \sigma$. Define a function $\phi \in C^\infty(\Omega)$ which assumes the value zero when $x \in \overline{\Omega} \setminus B_{r_0}$ and the value one when $x \in B_{r_0/2} \cap \overline{\Omega}$ and has range $[0,1]$. Then

$$\begin{aligned}\lim_{n\to\infty} \int_{B_{r_0}(x_0)\cap\overline{\Omega}} |\nabla u_n|^N \phi &= \int_{B_{r_0}(x_0)\cap\overline{\Omega}} \phi \, d\mu \\ &\leq \mu(B_{r_0}(x_0) \cap \overline{\Omega}) \leq \sigma.\end{aligned}$$

By the assumptions on $\sigma$, there exists some $q > 1$ such that

$$q\beta\sigma^{\frac{1}{N-1}} < \alpha_N$$

and hence

$$\int_{B_{r_0/2}(x_0)\cap\overline{\Omega}} |e(x, u_n(x))|^q \leq C.$$

Consequently, with all integration performed over the domain $B_{r_0/2}(x_0) \cap \overline{\Omega}$,

$$\int e(x, u_n)u_n - e(x, u)u = \int (e(x, u_n) - e(x, u))\, u + \int e(x, u_n)\,(u_n - u)\,.$$

We know that $e(x, u_n) \to e(x, u)$ in $L^1(\Omega)$ and so the first term on the right hand side tends to zero. Apply Hölder's inequality to the second term to reveal

$$\left|\int e(x, u_n)(u_n - u)\right| \leq \|e(x, u_n)\|_q \|u_n - u\|_{q'} \to 0.$$

Consequently $\int_{B_{r_0}(x_0)\cap\overline{\Omega}}(e(x, u_n)u_n - e(x, u)u) \to 0$. Since $K$ is a compact set, repeating the same procedure over a finite covering of balls gives the result.

**Assertion 2.**

$$\int_{\Omega_\epsilon} \left(|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u\right)(\nabla u_n - \nabla u) \to 0 \ \text{ as } \ n \to \infty$$

where $\epsilon$ is sufficiently small that $B(x_i, \epsilon) \cap B(x_j, \epsilon) = \emptyset$ for $i \neq j$ and

$$\Omega_\epsilon = \left\{x \in \overline{\Omega} : \|x - x_j\| \geq \epsilon\right\}.$$

That is, $\Omega_\epsilon$ consists of $\overline{\Omega}$ except for $m$ $\epsilon$-balls around $\{x_1, \ldots, x_m\}$.

Let $0 \leq \psi_\epsilon(x) \leq 1$ be a $C^\infty(\Omega)$ function set to be 1 on $\Omega_\epsilon$ and 0 on $\bigcup_{i=1}^m B(x_i, \epsilon/2)$.

From the Palais-Smale sequence (24):

$$\left|\int |\nabla u_n|^{N-2}\nabla u_n \nabla v - \int e(x, u_n)v - \int hv\right| \to 0.$$

Successively substitute $v = \psi_\epsilon u_n$ and $v = \psi_\epsilon u$, then

$$\begin{aligned}&\left[\int |\nabla u_n|^N\psi_\epsilon + |\nabla u_n|^{N-2}(\nabla u_n.\nabla\psi_\epsilon)u_n - \int e(x, u_n)u_n\psi_\epsilon - \int hu_n\psi_\epsilon\right]\\ &\qquad\qquad \leq \epsilon_n\|\psi_\epsilon u_n\|;\\ &\left[-\int |\nabla u_n|^{N-2}(\nabla u_n.\nabla u)\psi_\epsilon - |\nabla u_n|^{N-2}(\nabla u_n.\nabla\psi_\epsilon)u\right.\\ &\qquad\qquad \left. + \int e(x, u_n)u\psi_\epsilon + \int hu\psi_\epsilon\right] \leq \epsilon_n\|\psi_\epsilon u\|.\end{aligned}$$

Combine these by addition, then

$$\begin{aligned}\int \psi_\epsilon |\nabla u_n|^{N-2}\nabla u_n.(\nabla u - \nabla u_n) \le & \int |\nabla u_n|^{N-2}(\nabla u_n.\nabla\psi_\epsilon)(u-u_n) \\ & + \int \psi_\epsilon |\nabla u|^{N-2}\nabla u(\nabla u - \nabla u_n) \\ & + \int \psi_\epsilon e(x,u_n)(u_n - u) + \int h\psi_\epsilon(u_n-u) \\ & + \epsilon_n\|\psi_\epsilon u_n\| + \epsilon_n\|\psi_\epsilon u\|.\end{aligned} \tag{27}$$

Make use of the convexity of the map $v \mapsto |v|^N$ to establish that

$$\left(|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u\right)(\nabla u_n - \nabla u) \ge 0$$

and to prove that the left hand side in (27) is nonnegative. Estimate each of the integrals in (27) using the information that $u_n \rightharpoonup u_0$ in $W_0^{1,N}$ to show that

$$\int_{\Omega_\epsilon} \left(|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u\right)(\nabla u_n - \nabla u) \to 0 \quad \text{as} \quad n \to \infty.$$

Since $\epsilon$ is arbitrary,

$$\nabla u_n(x) \to \nabla u(x) \quad \text{a.e. in } \Omega$$

and using the boundedness of $(|\nabla u_n|^{N-2}\nabla u_n)$ in $(L^{\frac{N}{N-1}})^N$, we have for a subsequence

$$|\nabla u_n|^{N-2}\nabla u_n \rightharpoonup |\nabla u|^{N-2}\nabla u \quad \text{in} \quad \left(L^{\frac{N}{N-1}}(\Omega)\right)^N.$$

□

**Corollary 3.3.** *It follows from Lemma* 3.2 *that any Palais-Smale sequence for $I$ or $I^+$ is bounded and weakly convergent to a weak solution of* (1) *or* (3) *respectively.*

PROOF: Again the proof is identical for $I$ and $I^+$.

Suppose $\{u_n\}$ is a Palais-Smale sequence. Using the previous lemma and equation (24),

$$\int |\nabla u_0|^{N-2}\nabla u_0.\nabla w - \int e(x,u_0)w - \int hw = 0 \quad \text{for all} \quad w \in \mathcal{D}(\Omega)$$

and thus $u_0$ is a weak solution. Since $h(x) \not\equiv 0$, $u_0 \not\equiv 0$. □

**Remark 3.4.** *The case of $h \equiv 0$ is covered in* [13]. *There, convergence to a nontrivial solution relies upon the energy of the sequence remaining below a forbidden level,*

$$\frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

**Lemma 3.5.** *If $\{u_n\}$ is a (PS)-sequence for $I$ or $I^+$ at any level with*

$$\liminf_{n\to\infty} \|u_n\| < \left(\frac{\alpha_N}{\alpha_0}\right)^{\frac{N-1}{N}}$$

*then a subsequence converges strongly to a solution $u_0$.*

PROOF: Again the proof for $I$ and $I^+$ is constructed in an identical manner. Let $\{u_n\}$ be such a sequence. Extract a subsequence, again relabelled as $u_n$ such that $\lim_{n\to\infty}\|u_n\| = \liminf_{n\to\infty}\|u_n\|$. Corollary 3.3 establishes that $u_n$ converges weakly to $u_0$, a solution of (1). Let $u_n = u_0 + w_n$. Then $w_n \rightharpoonup 0$ in $W_0^{1,N}$ and $w_n \to 0$ in $L^t$ for all $t \geq 1$. By the Brezis-Lieb lemma [4], $\|u_n\|^N = \|u_0\|^N + \|w_n\|^N + o(1)$. Since $u_0 \in W_0^{1,N}$, Lemma 3.2 implies that $\int e(x,u_n)u_0 \to \int e(x,u_0)u_0$. This gives that

$$(I'(u_n),u_n) = (I'(u_0),u_0) + \|w_n\|^N - \int e(x,u_n)w_n + o(1).$$

Since $\lim\|u_n\| < (\frac{\alpha_N}{\alpha_0})^{\frac{N-1}{N}}$, choose $q > 1$ such that

$$\lim_{n\to\infty} q\alpha_0\|u_n\|^{\frac{N}{N-1}} < \alpha_N.$$

Now,

$$\begin{aligned}\int |e(x,u_n)|^q &\leq C\int \exp\left(q\alpha_0|u_n|^{\frac{N}{N-1}}\right)\\ &= C\int \exp\left(q\alpha_0\|u_n\|^{\frac{N}{N-1}}\left|\frac{u_n}{\|u_n\|}\right|^{\frac{N}{N-1}}\right)\\ &\leq C.\end{aligned}$$

Thus $\int e(x,u_n)w_n \leq \|e(x,u_n)\|_q\|w_n\|_{q'} \to 0$. Consequently $\|w_n\| \to 0$ and the result follows. □

A local semicontinuity result is expressed below.

**Lemma 3.6.** *For any fixed $\epsilon > 0$, let $B$ be the ball in $W_0^{1,N}(\Omega)$ centred at the origin with radius $(\frac{\alpha_N}{\alpha_0})^{\frac{N-1}{N}} - \epsilon$. The functionals $I(u)$ and $I^+(u)$ are lower semicontinuous on $B$.*

PROOF: Again the method of proof is the same for $I$ and $I^+$.

Let $\{u_n\} \subset B$. Then $u_n \rightharpoonup u_0 \in B$ and

$$I(u_n) - I(u_0) = \frac{1}{N}\|u_n\| - \frac{1}{N}\|u_0\| - \int E(x,u_n) + \int E(x,u_0) + o(1). \tag{28}$$

We now show that $E(x,u_n) \to E(x,u_0)$ in $L^1$ by invoking Lemma 3.1.

Equation (28) establishes that $\lim_{n\to\infty} \int E(x,u_n) < \infty$. From condition (10), there exists $\overline{E} > 0$ such that $E(x,u) \le \overline{E} + Me(x,u)$. Consequently,

$$\int E(x,u_n)u_n \le \int \overline{E}u_n + M\int e(x,u_n)u_n.$$

The first term is bounded as $u_n \to u_0$ in $L^1(\Omega)$.

Since $u_n$ is bounded, $I'(u_n)$ must be bounded. Consequently

$$(I'(u_n),u_n) = \|u_n\|^N - \int e(x,u_n)u_n - \int hu_n \le C.$$

But $\|u_n\|^N$ and $\int hu_n \le \|h\|_*\|u_n\|$ are both bounded and subsequently $\int e(x,u_n)u_n \le C$. Thus, Lemma 3.1 proves that $\int E(x,u_n) \to \int E(x,u_0)$. As a consequence, $I(u_n) - I(u_0) \ge 0$ by lower semicontinuity of norms. □

## 4. Generation of solutions

A solution can be obtained by local minimisation near the origin in $W_0^{1,N}(\Omega)$. To show the existence of this solution, we use Ekeland's variational principle ([9]). The number $c_0$ is defined in (19).

**Lemma 4.1.** *For $h(x) \in W^{-1,N'}$ with $0 < \|h\|_* \le h^*$, there exists a minimum type solution, $u_0$, to* (1) *at energy $c_0 < 0$. As $\|h\|_* \to 0$, $\|u_0\| \to 0$.*

Proof: Let $\rho_h$ be the radius of the ball from Lemma 2.1. Consulting (17) in the proof of Lemma 2.1, we see that $h^* > 0$ has been chosen sufficiently small that $\rho_h < (\frac{\alpha_N}{\alpha_0})^{\frac{N-1}{N}}$. It follows that $B(0,\rho_h) \subset W_0^{1,N}(\Omega)$ forms a complete metric space with metric $d(u_1,u_2) = \|u_1 - u_2\|$. On this set, $I$ is lower semicontinuous and bounded below. Ekeland's principle ensures the existence of a $(\mathrm{PS})_{c_0}$-sequence from a minimising sequence $\{u_n\}$ for $I$ in $B(0,\rho_h)$ in the same way as [6]. Each element of the sequence minimises

$$\inf\left\{I(u) + \delta_n\|u_n - u\|_{W_0^{1,N}} : u \in \overline{B(0,\rho_h)}\right\} \tag{29}$$

for some $0 < \delta_n \to 0$ as $n \to \infty$. By Lemma 2.2, $I(u^*) < 0$ for some $u^* \in B(0,\rho_h)$ so $I(u_n) \to c_0 < 0$. Condition (29) implies $I'(u_n) \to 0$ in $W^{-1,N'}$ providing the Palais-Smale sequence. Lemma 3.5 guarantees that this sequence converges strongly to the minimiser which must be a solution. □

**Lemma 4.2.** *Suppose that* (i) *$h(x) \ge 0$ almost everywhere and $e(x,u)$ satisfies* (12), *or* (ii) *$h(x)$ is indefinite in sign and $e(x,u)$ satisfies* (13). *There exists a number $h^{**} > 0$ such that the mountain pass geometry reveals a solution, $u_M$, to* (1) *when $\|h\|_* \le h^{**}$.*

Proof: Lemmas 2.3 and 2.7 verify that there exists some $\tilde{u} \in W_0^{1,N}(\Omega)$ such that $I(t\tilde{u}) < \frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1}$ for all $t \ge 0$ and $I(\overline{t}\tilde{u}) < 0$ for some large $\overline{t} > 0$. Lemma 2.1 guarantees that a mountain ridge exists.

Invoking the mountain pass theorem without a Palais-Smale condition ([3]) provides a Palais-Smale sequence. Although not required for this lemma, we incidentally remark that for suitably small $h^{**}$, the energy of the sequence lies below $c_0 + \frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1}$.

Corollary 3.3 assures that this (PS) sequence converges weakly to a weak solution of (1).

□

**Lemma 4.3.** *For suitably small $h^{**}$ the solutions derived in Lemmas 4.1 and 4.2 are distinct.*

PROOF: Let $\{u_n\}$ be the minimising sequence and $\{v_n\}$ be the mountain pass sequence, so that

$$\begin{array}{lll} u_n \rightharpoonup u_0 & \text{and} & v_n \rightharpoonup u_M \\ I(u_n) \to c_0 < 0 & \text{and} & I(v_n) \to c_M > 0 \\ (I'(u_n), u_n) \to 0 & \text{and} & (I'(v_n), v_n) \to 0. \end{array}$$

Suppose that $u_0 = u_M$. Then from Lemma 3.2

$$I(u_n) = \frac{1}{N}\|u_n\|^N - \int E(x, u_0) - \int hu_0 + o(1) \to c_0$$
$$I(v_n) = \frac{1}{N}\|v_n\|^N - \int E(x, u_0) - \int hu_0 + o(1) \to c_M$$

and subtracting one from the other, we have

$$\|u_n\|^N - \|v_n\|^N \to N(c_0 - c_M) < 0 \quad \text{as} \quad n \to \infty. \tag{30}$$

Since $u_n$ and $v_n$ are both Palais-Smale sequences,

$$\big(I'(u_n), u_n\big) = \int |\nabla u_n|^N - \int e(x, u_n)u_n - \int hu_n \to 0$$
$$\big(I'(v_n), v_n\big) = \int |\nabla v_n|^N - \int e(x, v_n)v_n - \int hv_n \to 0$$

to give

$$\begin{aligned} \Big(\|u_n\|^N - \|v_n\|^N\Big) &- \int [e(x, u_n)u_n - e(x, u_n)v_n + e(x, u_n)v_n - e(x, v_n)v_n] \\ &- \int [h(u_n - u_0) - h(v_n - u_0)] \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{31}$$

Since $h \in W^{-1,N'}$ and $u_n \rightharpoonup u_0$ and $v_n \rightharpoonup u_0$, the last term tends to zero.

The second term may be written:

$$\int e(x,u_n)(u_n - v_n) + \int [e(x,u_n) - e(x,v_n)]\, v_n.$$

We have derived that for $\|h\|_*$ in the range $(0,h^*)$, the minimising sequence $\{u_n\}$ must satisfy $\|u_n\| < (\frac{\alpha_N}{\alpha_0})^{\frac{N-1}{N}}$. Letting $q$ be slightly larger than 1 it follows that

$$\begin{aligned}\int |e(x,u_n)|^q &\le C \int \exp\left(q\alpha_0 |u_n|^{\frac{N}{N-1}}\right)\\ &= C\int \exp\left(q\alpha_0 \|u_n\|^{\frac{N}{N-1}} \left|\frac{u_n}{\|u_n\|}\right|^{\frac{N}{N-1}}\right)\\ &\le C.\end{aligned}$$

By the fact that $(u_n - v_n) \rightharpoonup 0$ in $W_0^{1,N}$,

$$\int e(x,u_n)(u_n - v_n) \le \|e(x,u_n)\|_q \|u_n - v_n\|_{q'} \le C\|u_n - v_n\|_{q'} \to 0.$$

It remains to show that

$$\int [e(x,u_n) - e(x,v_n)]\, v_n \to 0. \tag{32}$$

Let $v_n = u_0 + w_n$, so $w_n \rightharpoonup 0$. However, since $v_n$ is a mountain pass sequence, $v_n \not\to u_0$. Consequently, $v_n$ must concentrate and $\lim \|w_n\| > 0$. Now, (32) may be expressed as

$$\int [e(x,u_n) - e(x,v_n)]\, u_0 + \int [e(x,u_n) - e(x,v_n)]\, w_n \to 0.$$

Lemma 3.2 establishes that $e(x,u_n)$ and $e(x,v_n)$ both converge in $L^1(\Omega)$ to $e(x,u_0)$ and so the first term vanishes. Considering the second of these terms,

$$\int [e(x,u_n) - e(x,v_n)]\, w_n = \int e(x,u_n) w_n - \int e(x,v_n) w_n.$$

The minimising sequence $u_n$ has the property that $\|u_n\| < (\frac{\alpha_N}{\alpha_0})^{\frac{N-1}{N}}$. Consequently

$$\begin{aligned}\int e(x,u_n) w_n &\le \|e(x,u_n)\|_q \|w_n\|_{q'}\\ &\le \left[C\int \exp\left(q\alpha_0 \|u_n\|^{\frac{N}{N-1}} \left|\frac{u_n}{\|u_n\|}\right|^{\frac{N}{N-1}}\right)\right]^{\frac{1}{q}} \|w_n\|_{q'}\\ &\le C\|w_n\|_{q'} \to 0.\end{aligned}$$

We are now left with only the term $\int e(x,v_n)w_n$.

By Lemma 2.9, the value of $h^{**}$ is sufficiently small that we are guaranteed that for large $n$,

$$\begin{aligned} c_M - c_0 &= I(v_n) - I(u_n) + o(1) = \frac{1}{N}\|v_n\|^N - \frac{1}{N}\|u_n\|^N + o(1) \\ &= \frac{1}{N}\|v_n\|^N - \frac{1}{N}\|u_0\|^N + o(1) < \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{\frac{N-1}{N}}. \end{aligned}$$

Thus there exists $q > 1$ such that for large $n$, $\|v_n\|^N - \|u_0\|^N < (\frac{\alpha_N}{q\alpha_0})^{N-1}$. As a direct implication,

$$\begin{aligned} q^{N-1}\left(\frac{\alpha_0}{\alpha_N}\right)^{N-1} &< \frac{1}{\|v_n\|^N - \|u_0\|^N} \\ \Rightarrow q\alpha_0\|v_n\|^{\frac{N}{N-1}} &< \alpha_N\left[1 - \left\|\frac{u_0}{\|v_n\|}\right\|^N\right]^{\frac{-1}{N-1}}. \end{aligned} \tag{33}$$

Define $U_n = \frac{v_n}{\|v_n\|}$. Thus $\|U_n\| = 1$ and $U_n \rightharpoonup U_0 = \frac{u_0}{\lim\|v_n\|}$. We have deduced that $v_n$ concentrates and hence $\|U_0\| < 1$.

Now,

$$\int e(x,v_n)w_n \leq \|e(x,v_n)\|_q\|w_n\|_{q'}$$

but

$$\int |e(x,v_n)|^q \leq C\int \exp\left(q\alpha_0\|v_n\|^{\frac{N}{N-1}}\left|\frac{v_n}{\|v_n\|}\right|^{\frac{N}{N-1}}\right).$$

The right-hand side may be expressed as

$$C\int \exp\left(p\alpha_N|U_n|^{\frac{N}{N-1}}\right)$$

where (33) exposes that $p$ is within the range demanded by Theorem 1.3. As a consequence, $\|e(x,v_n)\|_q$ is bounded. Using the information that $\|w_n\|_{q'} \to 0$, it follows that $\int e(x,u_n)u_n - e(x,v_n)v_n \to 0$. Hence expression (31) gives that $\|u_n\|^N - \|v_n\|^N \to 0$. But this contradicts (30), and thus $u_0 \not\equiv u_M$ and the solutions are distinct. □

The proof to Theorem 1.1 now follows from Lemmas 4.1, 4.2 and 4.3.

## 5. Signs of solutions

Tarantello's results [16] deduced that for a similar problem, a positive perturbation gives rise to positive solutions. The technique from [15] will be implemented to attain a similar result.

Publication [13] claims that for $h \equiv 0$, solutions to (1) are nonnegative, but this is not proven. We confirm this result here, and as a consequence discover that a negative solution exists also. This occurs despite the fact that the nonlinearity $E(x,u)$ is not necessarily even in $u$.

**Theorem 5.1.** *Suppose $h \equiv 0$ and* (13) *holds. There exist at least one nonnegative and one nonpositive solution to* (1).

PROOF: Take the positive half of $E(x,u)$ and symmetrise. Define

$$\overline{E}(x,u) = \begin{cases} E(x,u) & \text{if } u \geq 0 \\ E(x,-u) & \text{if } u < 0. \end{cases}$$

Define $\overline{e}$ accordingly, and construct $\overline{I} = \frac{1}{N}\|u\|^N - \int \overline{E}(x,u)$. The even functional $\overline{I}$ satisfies the required geometry and convergence properties (Lemmas 2.1, 2.3, 2.4 and Remark 3.4), and so the mountain pass lemma without the (PS) condition exposes a nontrivial solution $\overline{u}$. It follows easily that $\overline{I}(\overline{u}) = \overline{I}(|\overline{u}|)$, and we may assume $\overline{u} = |\overline{u}| \geq 0$ is a solution. Since $I$ and $\overline{I}$ correspond when $\overline{u} \geq 0$, we have that $\overline{u}$ solves (1).

To locate a negative solution, define

$$\underline{E}(x,u) = \begin{cases} E(x,-u) & \text{if } u > 0 \\ E(x,u) & \text{if } u \leq 0. \end{cases} \tag{34}$$

With $\underline{e}$ and $\underline{I}$ defined accordingly, the pertinent geometry and convergence properties hold and a nontrivial solution $\underline{u}$ results. By the remarks above, $\underline{u} \geq 0$ in $\Omega$. For any $v \in C_0^\infty(\Omega)$,

$$(\underline{I}'(\underline{u}), v) = \int |\nabla \underline{u}|^{N-2} \nabla \underline{u} \nabla v - \int \underline{e}(x,\underline{u}) v$$

but since $\underline{u} \geq 0$, $\underline{e}(x,\underline{u}) = -e(x,-\underline{u})$. Consequently,

$$\int |\nabla(-\underline{u})|^{N-2} \nabla(-\underline{u}) \nabla v - \int e(x,-\underline{u}) v = 0$$

and so $-\underline{u}$ is a nonpositive solution. Since solutions are of opposing signs and neither is trivial they must be distinct. $\square$

**Lemma 5.2.** *If* (12) *holds and nonzero $h(x) \geq 0$, then the two derived solutions are positive.*

PROOF: We have shown that the functional $I^+(u)$ satisfies the pertinent geometry and convergence properties. Lemmas 4.1, 4.2 and 4.3 are applicable to $I(u)$ as well as to $I^+(u)$ and subsequently the functional $I^+$ elicits two critical points.

Let $u$ be a critical point of $I^+$. Decompose $u$ as $u = u^+ - u^-$ where $u^+ \geq 0$ and $u^- \geq 0$. Then

$$((I^+)'(u), u^-) = \int |\nabla u|^{N-2} \nabla u . \nabla u^- - \int e^+(x,u) u^- - \int h u^-.$$

However, $e^+(x,u)u^-(x) = 0$ almost everywhere and hence

$$-\|u^-\|^N - \int h u^- = 0.$$

But $h(x)u^-(x) \geq 0$ almost everywhere, and thus $\|u^-\| = 0$. Consequently $u(x) \geq 0$ on $\Omega$. $\square$

**Lemma 5.3.** *If* (13) *holds and nonzero $h(x) \leq 0$ on $\Omega$, then there exist at least two negative solutions.*

Proof: Assume the definition for $\underline{E}$ from (34). Define $\underline{I}(u) = \frac{1}{N}\|u\|^N - \int \underline{E}(x,u) + \int hu$. Since $-h(x) \geq 0$, the conditions are fulfilled to implement Lemma 5.2 exposing two nonnegative solutions to $\underline{I}'(u) = 0$. Considering one such solution, $\underline{u}$, and recalling the construction of $\underline{E}(x,u)$, we have that $\underline{e}(x,\underline{u}) = -e(x,-\underline{u})$ and so

$$\begin{aligned}
-\big(\underline{I}'(\underline{u}),v\big) &= -\int |\nabla \underline{u}|^{N-2}\nabla \underline{u}\nabla v + \int \underline{e}(x,\underline{u})v - \int hv \\
&= \int |-\nabla \underline{u}|^{N-2}(-\nabla \underline{u})\nabla v - \int e(x,-\underline{u})v - \int hv \\
&= \big(I'(-\underline{u}),v\big) = 0.
\end{aligned}$$

□

**Remark 5.4.** *For the development of positive solutions with $h \geq 0$ in Theorem* 5.1 *and Lemma* 5.2*, condition* (12) *will suffice in place of* (13)*. Further,* (10) *and* (11) *may be relaxed to:*
*There exists $R > 0$ and $M > 0$ such that for all $u \geq R$ and $x \in \Omega$,*

$$0 < E(x,u) \leq Me(x,u) \quad \text{and}$$
$$\limsup_{u\to 0+} \frac{NE(x,u)}{|u|^N} < \lambda_1$$

*respectively.*

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