Hopf algebra structure $H^{\sigma-R}$ with two sided invertible 2-cocycle

Wang Shuanhong, Wang Dingguo

Abstract. In this paper, we study the $H^{\sigma-R}$ type Hopf algebras and present its braided and quasitriangular Hopf algebra structure. This generalizes well-known results on H^{σ} and H^{R} type Hopf algebras. Finally, the classification of $H^{\sigma-R}$ type Hopf algebras is given.

Keywords: Hopf algebra, 2-cocycle, braided Hopf algebra

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The cocycle deformation H^{σ} of a bialgebra H was introduced and the Hopf algebra structure and the braided bialgebra structure of H^{σ} were given by Doi [1]. Doi and Takeuchi [2] showed that the Drinfeld double $D(H)$ for a finite-dimensional Hopf algebra H is a cocycle deformation of $(H^*)^{cop} \otimes H$. This suggests that cocycle deformations will play an important role in quantum group theory. The bialgebra H^{σ} has the same underlying coalgebra H, so that they have the same corepresentation theory. In fact, these are well known classes of 2-cocycles which appear as the duals of the Reshetikhin-type twists H^R ([8]) which originate from particular solutions of the QYBE. The Reshetikhin-type twist H^R also plays an important role in quantum group theory. The bialgebra H^R has the same underlying algebra H, so that they have the same representation theory. In this paper we consider the mixed type twist $H^{\sigma-R}$ and give Hopf algebra structure of $H^{\sigma-R}$, and its braided and quasitriangular Hopf algebra structure are also established. This generalizes well-known results on H^{σ} type Hopf algebras and H^R type Hopf algebras. Finally, the classification of $H^{\sigma-R}$ type Hopf algebras is given.

Throughout this paper, we refer to [3] and [5] for full details on Hopf algebras, and in particular we use the sigma notation: $\Delta(h) = \sum h_1 \otimes h_2$.

Let H be a $\sigma-$Hopf algebra, that is, there is a bilinear map $\sigma : H \times H \longrightarrow k$ such that for all $x, y, z \in H$ we have

$$\sum \sigma(x_1, y_1)\sigma(x_2 y_2, z) = \sum \sigma(y_1, z_1)\sigma(x, y_2 z_2), \tag{1}$$

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$$\sigma(x,1)=\sigma(1,x)=\varepsilon(x)1. \tag{2}$$

If σ is invertible, its inverse is denoted by $\sigma^{-1}: H\times H\longrightarrow k$, then by [1, Theorem 1.6(a)], we have that for all $x,y,z\in H$

$$\sum\sigma^{-1}(x_1y_1,z)\sigma^{-1}(x_2,y_2)=\sum\sigma^{-1}(x,y_1z_1)\sigma^{-1}(y_2,z_2), \tag{3}$$

$$\sigma^{-1}(x,1)=\sigma^{-1}(1,x)=\varepsilon(x)1. \tag{4}$$

By [1, Theorem 1.6(b)], H has a new algebra structure:

$$h\odot l=\sum\sigma(h_1,l_1)h_2l_2\sigma^{-1}(h_3,l_3)\quad\text{for all}\quad h,l\in H \tag{A}$$

and $(H,\odot,\Delta_H)$ is a Hopf algebra.

Dually, let H be a $R-$Hopf algebra, that is, there is an element $R=\sum R^{(1)}\otimes R^{(2)}\in H\otimes H$ which satisfies the following conditions ($r=R$):

$$\sum R^{(1)}{r^{(1)}}_1\otimes R^{(2)}{r^{(1)}}_2\otimes r^{(2)}=\sum r^{(1)}\otimes R^{(1)}{r^{(2)}}_1\otimes R^{(2)}{r^{(2)}}_2, \tag{1'}$$

$$\sum\varepsilon(R^{(1)})R^{(2)}=\sum R^{(1)}\varepsilon(R^{(2)})=1. \tag{2'}$$

If R is invertible with inverse $U=\sum U^{(1)}\otimes U^{(2)}\in H\otimes H$, we have by [8]: $U=u$

$$\sum U^{(1)}\otimes{U^{(2)}}_1u^{(1)}\otimes{U^{(2)}}_2u^{(2)}=\sum{U^{(1)}}_1u^{(1)}\otimes{U^{(1)}}_2u^{(2)}\otimes U^{(2)}, \tag{3'}$$

$$\sum\varepsilon(U^{(1)})U^{(2)}=\sum U^{(1)}\varepsilon(U^{(2)})=1. \tag{4'}$$

By [3], H has a new comultiplication:

$$\widetilde{\Delta}(h)=\sum R^{(1)}h_1U^{(1)}\otimes R^{(2)}h_2U^{(2)}\quad\text{for all}\quad h\in H \tag{B}$$

and $(H,m_H,\widetilde{\Delta})$ is a Hopf algebra.

Remark. Condition $(1')$ was first obtained by setting $C=k$ in the crossed co-product $C\times_\sigma H$ (see [7]), then we found that condition $(1')$ is dual to condition (1).

1. Hopf algebra structure of type $H^{\sigma-R}$

In this section, we assume that H is $\sigma-$Hopf and $R-$Hopf algebra. When σ and R are compatible, we prove that $(H,\odot,\widetilde{\Delta})$ is a Hopf algebra, and denote it by $H^{\sigma-R}$. In the following we always assume that σ has convolution inverse and R has inverse $U=\sum U^{(1)}\otimes U^{(2)}\in H\otimes H$.

Definition 1.1. *Let H be a $\sigma-$Hopf and $R-$Hopf algebra. H will be called a $\sigma-R$ compatible Hopf algebra, if σ and R satisfy the following conditions: $R=\widetilde{R}=r=\widetilde{r}$, $h,l\in H$*

- (i) $\sum\sigma(R^{(1)}hr^{(1)},\widetilde{R}^{(1)}l\widetilde{r}^{(1)})R^{(2)}\otimes r^{(2)}\otimes\widetilde{R}^{(2)}\otimes\widetilde{r}^{(2)}=\sigma(h,l)(1\otimes1\otimes1\otimes1);$
- (ii) $\sum\sigma(R^{(2)}hr^{(2)},\widetilde{R}^{(2)}l\widetilde{r}^{(2)})R^{(1)}\otimes r^{(1)}\otimes\widetilde{R}^{(1)}\otimes\widetilde{r}^{(1)}=\sigma(h,l)(1\otimes1\otimes1\otimes1);$
- (iii) $\sum\sigma(h,R^{(1)}{}_2)R^{(1)}{}_1\otimes R^{(2)}=\sum\sigma(R^{(1)}{}_2,h)R^{(1)}{}_1\otimes R^{(2)}=\sum(R^{(1)}\otimes R^{(2)})\varepsilon(h);$
- (iv) $\sum\sigma^{-1}(R^{(1)}h,r^{(1)}l)R^{(2)}\otimes r^{(2)}=\sum\sigma^{-1}(R^{(2)}h,r^{(2)}l)R^{(1)}\otimes r^{(1)}=\sum\sigma(h,l)(1\otimes1).$

Now we give some properties of $\sigma-R$ compatible Hopf algebras:

Proposition 1.2. *Let H be an $\sigma-R$ compatible Hopf algebra, then*

- (a) $\sum R^{(1)}\otimes R^{(1)}{}_1\sigma(R^{(2)}{}_2,h)=\sum R^{(1)}\otimes R^{(2)}{}_1\sigma(h,R^{(2)}{}_2)=\sum(R^{(1)}\otimes R^{(2)})\varepsilon(h);$
- (b) $\sum\sigma(R^{(1)}{}_1,l)R^{(1)}{}_2\otimes R^{(2)}=\sum\sigma(l,R^{(1)}{}_1)R^{(1)}{}_2\otimes R^{(2)}=\sum(R^{(1)}\otimes R^{(2)})\varepsilon(l);$
- (c) $\sum\sigma(h,R^{(2)}{}_1)R^{(2)}{}_2\otimes R^{(1)}=\sum\sigma(R^{(2)}{}_1,h)R^{(2)}{}_2\otimes R^{(1)}=\sum(R^{(2)}\otimes R^{(1)})\varepsilon(h);$
- (d) $\sum\sigma(U^{(1)}hu^{(1)},\widetilde{U}^{(1)}l\widetilde{u}^{(1)})U^{(2)}\otimes u^{(2)}\otimes\widetilde{U}^{(2)}\otimes\widetilde{u}^{(2)}=\sigma(h,l)(1\otimes1\otimes1\otimes1);$
- (e) $\sum\sigma(U^{(2)}hu^{(2)},\widetilde{U}^{(2)}l\widetilde{u}^{(2)})U^{(1)}\otimes u^{(1)}\otimes\widetilde{U}^{(1)}\otimes\widetilde{u}^{(1)}=\sigma(h,l)(1\otimes1\otimes1\otimes1);$
- (f) $\sum\sigma(U^{(1)}{}_1,l)U^{(1)}{}_2\otimes U^{(2)}=\sum\sigma(l,U^{(1)}{}_1)U^{(1)}{}_2\otimes U^{(2)}=\sum(U^{(1)}\otimes U^{(2)})\varepsilon(l);$
- (g) $\sum U^{(1)}\otimes U^{(2)}{}_1\sigma(U^{(2)}{}_2,l)=\sum U^{(1)}\otimes U^{(2)}{}_1\sigma(l,U^{(2)}{}_2)=\sum(U^{(1)}\otimes U^{(2)})\varepsilon(l).$

PROOF: (a)

$$
\begin{aligned}
&\sum R^{(1)}\otimes R^{(1)}{}_1\sigma(R^{(2)}{}_2,h)\\
\overset{(ii)+(2')}{=}&\sum R^{(1)}\otimes r^{(1)}R^{(2)}{}_1\sigma(r^{(2)}R^{(2)}{}_2,h)\\
\overset{(1')}{=}&\sum r^{(1)}R^{(1)}{}_1\otimes r^{(2)}R^{(1)}{}_2\sigma(R^{(2)},h)\\
\overset{(ii)+(2')}{=}&\sum R^{(1)}\otimes R^{(2)}\sigma(1,h)\\
\overset{(2)}{=}&\sum(R^{(1)}\otimes R^{(2)})\varepsilon(h).
\end{aligned}
$$

Similarly, we can prove the second formula of (a), (b) and (c).

By (i) and (ii) of Definition 1.1, and $RU = UR = 1 \otimes 1$, we can prove (d) and (e).

$$\begin{aligned}
&\sum \sigma(U^{(1)}{}_1, l)U^{(1)}{}_2 \otimes U^{(2)} \\
\overset{(d)+(4')}{=}&\sum \sigma(U^{(1)}{}_1 u^{(1)}, l)U^{(1)}{}_2 u^{(2)} \otimes U^{(2)} \\
\overset{(3')}{=}&\sum \sigma(U^{(1)}, l)U^{(2)}{}_1 u^{(1)} \otimes U^{(2)}{}_2 u^{(2)} \\
\overset{(d)+(4')}{=}&\sum \sigma(1, l)U^{(1)} \otimes U^{(2)} \\
\overset{(2)}{=}&\sum (U^{(1)} \otimes U^{(2)})\varepsilon(l).
\end{aligned}$$

Analogously, we can prove the second formula of (f) and (g). □

The following proposition can be proved by direct computation.

Proposition 1.3. *Let H be an $\sigma - R$ compatible Hopf algebra, then we have*

(h) $\sum R^{(1)} \otimes R^{(2)}{}_1\sigma^{-1}(R^{(2)}{}_2, l) = \sum R^{(1)} \otimes R^{(2)}{}_1\sigma^{-1}(l, R^{(2)}{}_2) = \sum (R^{(1)} \otimes R^{(2)})\varepsilon(l)$;

(i) $\sum R^{(1)}{}_1\sigma^{-1}(R^{(1)}{}_2, l) \otimes R^{(2)} = \sum R^{(1)}{}_1\sigma^{-1}(l, R^{(1)}{}_2) \otimes R^{(2)} = \sum (R^{(1)} \otimes R^{(2)})\varepsilon(l)$;

(j) $\sum \sigma^{-1}(R^{(1)}{}_1, l)R^{(1)}{}_2 \otimes R^{(2)} = \sum \sigma^{-1}(l, R^{(1)}{}_1)R^{(1)}{}_2 \otimes R^{(2)} = \sum (R^{(1)} \otimes R^{(2)})\varepsilon(l)$;

(k) $\sum \sigma^{-1}(h, R^{(2)}{}_1)R^{(2)}{}_2 \otimes R^{(1)} = \sum \sigma^{-1}(R^{(2)}{}_1, h)R^{(2)}{}_2 \otimes R^{(1)} = \sum (R^{(2)} \otimes R^{(1)})\varepsilon(h)$;

(l) $\sum U^{(1)} \otimes U^{(2)}{}_1\sigma^{-1}(U^{(2)}{}_2, l) = \sum U^{(1)} \otimes U^{(2)}{}_1\sigma^{-1}(l, U^{(2)}{}_2) = \sum (U^{(1)} \otimes U^{(2)})\varepsilon(l)$;

(m) $\sum \sigma^{-1}(U^{(1)}{}_1, l)U^{(1)}{}_2 \otimes U^{(2)} = \sum \sigma^{-1}(l, U^{(1)}{}_1)U^{(1)}{}_2 \otimes U^{(2)} = \sum (U^{(1)} \otimes U^{(2)})\varepsilon(l)$;

(n) $\sum \sigma^{-1}(R^{(2)}, l)R^{(1)} = \sum \sigma^{-1}(l, R^{(2)})R^{(1)} = \varepsilon(l)$;

(o) $\sum \sigma^{-1}(R^{(1)}, l)R^{(2)} = \sum \sigma^{-1}(l, R^{(1)})R^{(2)} = \varepsilon(l)$;

(p) $\sum \sigma^{-1}(U^{(2)}, l)U^{(1)} = \sum \sigma^{-1}(l, U^{(2)})U^{(1)} = \varepsilon(l)$;

(q) $\sum \sigma^{-1}(U^{(1)}, l)U^{(2)} = \sum \sigma^{-1}(l, U^{(1)})U^{(2)} = \varepsilon(l)$;

(r) $\sum \sigma^{-1}(U^{(1)}h, l)U^{(2)} = \sum \sigma^{-1}(hU^{(1)}, l)U^{(2)} = \sigma^{-1}(h, l)$;

(s) $\sum \sigma^{-1}(h, lU^{(2)})U^{(1)} = \sum \sigma^{-1}(hU^{(2)}, l)U^{(1)} = \sigma^{-1}(h, l)$.

Theorem 1.4. *Let H be an $\sigma - R$ compatible bialgebra, then $(H, \odot, \widetilde{\Delta})$ is a Hopf algebra which will be denoted by $H^{\sigma - R}$.*

PROOF: Clearly, $\widetilde{\Delta}(1)=1\otimes 1$. For all $h,l\in H$, we have

$$
\begin{aligned}
&\widetilde{\Delta}(h)\widetilde{\Delta}(l)\\
\overset{(B)}{=}&\sum(R^{(1)}h_1U^{(1)})\odot(r^{(1)}l_1u^{(1)})\otimes(R^{(2)}h_2U^{(2)})\odot(r^{(2)}l_2u^{(2)})\\
\overset{(A)}{=}&\sum\sigma(R^{(1)}{}_1h_1U^{(1)}{}_1,r^{(1)}{}_1l_1u^{(1)}{}_1)(R^{(1)}{}_2h_2U^{(1)}{}_2r^{(1)}{}_2l_2u^{(1)}{}_2)\\
&\sigma^{-1}(R^{(1)}{}_3h_3U^{(1)}{}_3,r^{(1)}{}_3l_3u^{(1)}{}_3)\\
&\otimes\sigma(R^{(2)}{}_1h_4U^{(2)}{}_1,r^{(2)}{}_1l_4u^{(2)}{}_1)(R^{(2)}{}_2h_5U^{(2)}{}_2r^{(2)}{}_2l_5u^{(2)}{}_2)\\
&\sigma^{-1}(R^{(2)}{}_3h_6U^{(2)}{}_3,r^{(2)}{}_3l_6u^{(2)}{}_3)\\
\overset{(b)}{=}&\sum\sigma(r^{(1)}{}_1,l_1U^{(1)}{}_1)\\
&\sigma(R^{(1)}{}_1h_1U^{(1)}{}_1,r^{(1)}{}_2l_2u^{(1)}{}_2)(R^{(1)}{}_2h_2U^{(1)}{}_2r^{(1)}{}_2l_2u^{(1)}{}_2)\\
&\sigma^{-1}(R^{(1)}{}_3h_3U^{(1)}{}_3,r^{(1)}{}_3l_3u^{(1)}{}_3)\\
&\otimes\sigma(R^{(2)}{}_1h_4U^{(2)}{}_1,r^{(2)}{}_1l_4u^{(2)}{}_1)(R^{(2)}{}_2h_5U^{(2)}{}_2r^{(2)}{}_2l_5u^{(2)}{}_2)\\
&\sigma^{-1}(R^{(2)}{}_3h_6U^{(2)}{}_3,r^{(2)}{}_3l_6u^{(2)}{}_3)\\
\overset{(1)}{=}&\sum\sigma(R^{(1)}{}_1h_1U^{(1)}{}_1,r^{(1)}{}_1)\\
&\sigma(R^{(1)}{}_2h_2U^{(1)}{}_2r^{(1)}{}_2,l_1u^{(1)}{}_1)(R^{(1)}{}_3h_3U^{(1)}{}_3r^{(1)}{}_3l_2u^{(1)}{}_2)\\
&\sigma^{-1}(R^{(1)}{}_4h_4U^{(1)}{}_4,r^{(1)}{}_4l_3u^{(1)}{}_3)\\
&\otimes\sigma(R^{(2)}{}_1h_5U^{(2)}{}_1,r^{(2)}{}_1l_4u^{(2)}{}_1)(R^{(2)}{}_2h_6U^{(2)}{}_2r^{(2)}{}_2l_5u^{(2)}{}_2)\\
&\sigma^{-1}(R^{(2)}{}_3h_7U^{(2)}{}_3,r^{(2)}{}_3l_6u^{(2)}{}_3)\\
\overset{(b)}{=}&\sum\sigma(R^{(1)}{}_1h_1U^{(1)}{}_1r^{(1)}{}_1,l_1u^{(1)}{}_1)(R^{(1)}{}_2h_2U^{(1)}{}_2r^{(1)}{}_2l_2u^{(1)}{}_2)\\
&\sigma^{-1}(R^{(1)}{}_3h_3U^{(1)}{}_3,r^{(1)}{}_3l_3u^{(1)}{}_3)\\
&\otimes\sigma(R^{(2)}{}_1h_4U^{(2)}{}_1,r^{(2)}{}_1l_4u^{(2)}{}_1)(R^{(2)}{}_2h_5U^{(2)}{}_2r^{(2)}{}_2l_5u^{(2)}{}_2)\\
&\sigma^{-1}(R^{(2)}{}_3h_6U^{(2)}{}_3,r^{(2)}{}_3l_6u^{(2)}{}_3)\\
\overset{(I)}{=}&\sum\sigma(R^{(1)}{}_1h_1U^{(1)}{}_1r^{(1)}{}_1,l_1u^{(1)}{}_1)(R^{(1)}{}_2h_2U^{(1)}{}_2r^{(1)}{}_2l_2u^{(1)}{}_2)\\
&\sigma^{-1}(R^{(1)}{}_3h_3U^{(1)}{}_3,r^{(1)}{}_3l_3u^{(1)}{}_3)\sigma^{-1}(r^{(1)}{}_4,l_4u^{(1)}{}_4)\\
&\otimes\sigma(R^{(2)}{}_1h_4U^{(2)}{}_1,r^{(2)}{}_1l_5u^{(2)}{}_1)(R^{(2)}{}_2h_5U^{(2)}{}_2r^{(2)}{}_2l_5u^{(2)}{}_2)\\
&\sigma^{-1}(R^{(2)}{}_3l_6U^{(2)}{}_3,r^{(2)}{}_3l_7u^{(2)}{}_3)\\
\overset{(3)}{=}&\sum\sigma(R^{(1)}{}_1h_1U^{(1)}{}_1r^{(1)}{}_1,l_1u^{(1)}{}_1)(R^{(1)}{}_2h_2U^{(1)}{}_2r^{(1)}{}_2l_2u^{(1)}{}_2)\\
&\sigma^{-1}(R^{(1)}{}_3h_3U^{(1)}{}_3r^{(1)}{}_3,l_3u^{(1)}{}_3)\sigma^{-1}(r^{(1)}{}_4h_4u^{(1)}{}_4,r^{(1)}{}_4)
\end{aligned}
$$

$$
\begin{aligned}
&\otimes \sigma(R^{(2)}{}_1h_5U^{(2)}{}_1, r^{(2)}{}_1l_4u^{(2)}{}_1)(R^{(2)}{}_2h_6U^{(2)}{}_2r^{(2)}{}_2l_5u^{(2)}{}_2)\\
&\sigma^{-1}(R^{(2)}{}_3h_7U^{(2)}{}_3, r^{(2)}{}_3l_6u^{(2)}{}_3)\\
\overset{(I)}{=}&\sum \sigma(R^{(1)}{}_1h_1U^{(1)}{}_1r^{(1)}{}_1, l_1u^{(1)}{}_1)(R^{(1)}{}_2h_2U^{(1)}{}_2r^{(1)}{}_2l_2u^{(1)}{}_2)\\
&\sigma^{-1}(R^{(1)}{}_3h_3U^{(1)}{}_3r^{(1)}{}_3, l_3u^{(1)}{}_3)\\
&\otimes \sigma(R^{(2)}{}_1h_4U^{(2)}{}_1, r^{(2)}{}_1l_4u^{(2)}{}_1)(R^{(2)}{}_2h_5U^{(2)}{}_2r^{(2)}{}_2l_5u^{(2)}{}_2)\\
&\sigma^{-1}(R^{(2)}{}_3h_6U^{(2)}{}_3, r^{(2)}{}_3l_6u^{(2)}{}_3)\\
\overset{(c)}{=}&\sum \sigma(R^{(1)}{}_1h_1U^{(1)}{}_1r^{(1)}{}_1, l_1u^{(1)}{}_1)(R^{(1)}{}_2h_2U^{(1)}{}_2r^{(1)}{}_2l_2u^{(1)}{}_2)\\
&\sigma^{-1}(R^{(1)}{}_3h_3U^{(1)}{}_3r^{(1)}{}_3, l_3u^{(1)}{}_3)\otimes \sigma(r^{(2)}{}_1, l_4u^{(2)}{}_1)\\
&\sigma(R^{(2)}{}_1h_4U^{(2)}{}_1, r^{(2)}{}_2l_5u^{(2)}{}_2)(R^{(2)}{}_2h_5U^{(2)}{}_2r^{(2)}{}_3l_6u^{(2)}{}_3)\\
&\sigma^{-1}(R^{(2)}{}_3h_6U^{(2)}{}_3, r^{(2)}{}_4l_7u^{(2)}{}_4)\\
\overset{(1)}{=}&\sum \sigma(R^{(1)}{}_1h_1U^{(1)}{}_1r^{(1)}{}_1, l_1u^{(1)}{}_1)(R^{(1)}{}_2h_2U^{(1)}{}_2r^{(1)}{}_2l_2u^{(1)}{}_2)\\
&\sigma^{-1}(R^{(1)}{}_3h_3U^{(1)}{}_3r^{(1)}{}_3, l_3u^{(1)}{}_3)\\
&\otimes \sigma(R^{(2)}{}_1h_4U^{(2)}{}_1, r^{(2)}{}_1)\\
&\sigma(R^{(2)}{}_2h_5U^{(2)}{}_2r^{(2)}{}_2, l_4u^{(2)}{}_1)(R^{(2)}{}_3h_6U^{(2)}{}_3r^{(2)}{}_3l_5u^{(2)}{}_2)\\
&\sigma^{-1}(R^{(2)}{}_4h_7U^{(2)}{}_4, r^{(2)}{}_4l_6u^{(2)}{}_3)\\
\overset{(h)+(c)}{=}&\sum \sigma(R^{(1)}{}_1h_1U^{(1)}{}_1r^{(1)}{}_1, l_1u^{(1)}{}_1)(R^{(1)}{}_2h_2U^{(1)}{}_2r^{(1)}{}_2l_2u^{(1)}{}_2)\\
&\sigma^{-1}(R^{(1)}{}_3h_3U^{(1)}{}_3r^{(1)}{}_3, l_3u^{(1)}{}_3)\\
&\otimes \sigma(R^{(2)}{}_1h_4U^{(2)}{}_1r^{(2)}{}_1, l_4u^{(2)}{}_1)(R^{(2)}{}_2h_5U^{(2)}{}_2r^{(2)}{}_2l_5u^{(2)}{}_2)\\
&\sigma^{-1}(R^{(2)}{}_3h_6U^{(2)}{}_3, r^{(2)}{}_3l_6u^{(2)}{}_3)\sigma^{-1}(R^{(2)}{}_4, l_7u^{(2)}{}_4)\\
\overset{(3)+(h)}{=}&\sum \sigma(R^{(1)}{}_1h_1U^{(1)}{}_1r^{(1)}{}_1, l_1u^{(1)}{}_1)(R^{(1)}{}_2h_2U^{(1)}{}_2r^{(1)}{}_2l_2u^{(1)}{}_2)\\
&\sigma^{-1}(R^{(1)}{}_3h_3U^{(1)}{}_3r^{(1)}{}_3, l_3u^{(1)}{}_3)\\
&\otimes \sigma(R^{(2)}{}_1h_4U^{(2)}{}_1r^{(2)}{}_1, l_4u^{(2)}{}_1)(R^{(2)}{}_2h_5U^{(2)}{}_2r^{(2)}{}_2l_5u^{(2)}{}_2)\\
&\sigma^{-1}(R^{(2)}{}_3h_6U^{(2)}{}_3r^{(2)}{}_3, l_6u^{(2)}{}_3)\\
\overset{\text{R invertible}}{=}&\sum \sigma(R^{(1)}{}_1h_1, l_1U^{(1)}{}_1)(R^{(1)}{}_2h_2l_2u^{(1)}{}_2)\sigma^{-1}(R^{(1)}{}_3h_3, l_3u^{(1)}{}_3)\\
&\otimes \sigma(R^{(2)}{}_1h_4, l_4U^{(2)}{}_1)(R^{(2)}{}_2h_5l_5u^{(2)}{}_2)\sigma^{-1}(R^{(2)}{}_3h_6, l_6u^{(2)}{}_3)\\
\overset{(i)}{=}&\sum \sigma(\tilde{R}^{(1)}R^{(1)}{}_1h_1, l_1u^{(1)}{}_1)(\tilde{R}^{(2)}_1R^{(1)}{}_2h_2l_2u^{(1)}{}_2)\\
&\sigma^{-1}(\tilde{R}^{(2)}_2R^{(1)}{}_3h_3, l_3u^{(1)}{}_3)
\end{aligned}
$$

$$
\begin{aligned}
&\otimes \sigma(R^{(2)}{}_3h_4, l_4U^{(2)}{}_1)(R^{(2)}{}_4h_5l_5u^{(1)}{}_2)\sigma^{-1}(R^{(2)}{}_5h_6, l_6u^{(2)}{}_3)\\
\overset{(1')}{=}&\sum \sigma(R^{(1)}h_1, l_1u^{(1)}{}_1)(\tilde{R}^{(1)}_1R^{(2)}{}_1h_2l_2u^{(1)}{}_2)\\
&\sigma^{-1}(\tilde{R}^{(1)}_2R^{(2)}{}_2h_3, l_3u^{(1)}{}_3)\\
&\otimes \sigma(\tilde{R}^{(2)}_1R^{(2)}{}_3h_4, l_4U^{(2)}{}_1)(\tilde{R}^{(2)}_2R^{(2)}{}_4h_5l_5u^{(1)}{}_2)\\
&\sigma^{-1}(\tilde{R}^{(2)}_3R^{(2)}{}_5h_6, l_6u^{(2)}{}_3)\\
\overset{(i)}{=}&\sum \sigma(h_1, l_1u^{(1)}{}_1)(\tilde{R}^{(1)}_1h_2l_2u^{(1)}{}_2)\sigma^{-1}(\tilde{R}^{(1)}_2h_3, l_3u^{(1)}{}_3)\\
&\otimes \sigma(\tilde{R}^{(2)}_1h_4, l_4u^{(2)}{}_1)(\tilde{R}^{(2)}_2h_5l_5u^{(1)}{}_2)\sigma^{-1}(\tilde{R}^{(2)}_3h_6, l_6u^{(2)}{}_3)\\
\overset{(d)}{=}&\sum \sigma(h_1, l_1u^{(1)}{}_1\tilde{u}^{(1)})(\tilde{R}^{(1)}_1h_2l_2u^{(1)}{}_2\tilde{u}^{(2)}_1)\sigma^{-1}(\tilde{R}^{(1)}_2h_3, l_3u^{(1)}{}_3\tilde{u}^{(2)}_2)\\
&\otimes \sigma(\tilde{R}^{(2)}_1h_4, l_4u^{(2)}{}_1)(\tilde{R}^{(2)}_2h_5l_5u^{(2)}{}_2)\sigma^{-1}(\tilde{R}^{(2)}_3h_6, l_6u^{(2)}{}_3)\\
\overset{(3')}{=}&\sum \sigma(h_1, l_1u^{(1)})(\tilde{R}^{(1)}_1h_2l_2u^{(2)}{}_1\tilde{u}^{(1)}_1)\sigma^{-1}(\tilde{R}^{(1)}_2h_3, l_3u^{(2)}{}_2\tilde{u}^{(1)}_2)\\
&\otimes \sigma(\tilde{R}^{(2)}_1h_4, l_4u^{(2)}{}_2\tilde{u}^{(2)}_1)(\tilde{R}^{(2)}_2h_5l_6u^{(2)}{}_3\tilde{u}^{(2)}_2)\sigma^{-1}(\tilde{R}^{(2)}_3h_6, l_6u^{(2)}{}_4\tilde{u}^{(2)}_3)\\
\overset{(d)}{=}&\sum \sigma(h_1, l_1)(\tilde{R}^{(1)}_1h_2l_2\tilde{u}^{(1)}{}_1)\sigma^{-1}(\tilde{R}^{(1)}_2h_3, l_3\tilde{u}^{(1)}_2)\\
&\otimes \sigma(\tilde{R}^{(2)}_1h_4, l_4\tilde{u}^{(2)}_1)(\tilde{R}^{(2)}_2h_5l_5\tilde{u}^{(2)}_2)\sigma^{-1}(\tilde{R}^{(2)}_3h_6, l_6\tilde{u}^{(2)}_3)\\
\overset{(n)}{=}&\sum \sigma(h_1, l_1)(R^{(1)}\tilde{R}^{(1)}_1h_2l_2\tilde{u}^{(1)}{}_1)\\
&\sigma^{-1}(R^{(2)}, \tilde{R}^{(1)}_2h_3l_3\tilde{u}^{(1)}_2)\sigma^{-1}(\tilde{R}^{(1)}_3h_4, l_4\tilde{u}^{(1)}_3)\\
&\otimes \sigma(\tilde{R}^{(2)}_1h_5, l_5\tilde{u}^{(2)}{}_1)(\tilde{R}^{(2)}_2h_6l_6\tilde{u}^{(2)}_2)\sigma^{-1}(\tilde{R}^{(2)}_3h_7, l_7\tilde{u}^{(2)}_3)\\
\overset{(3)}{=}&\sum \sigma(h_1, l_1)(R^{(1)}\tilde{R}^{(1)}_1h_2l_2\tilde{u}^{(1)}{}_1)\\
&\sigma^{-1}(R^{(2)}_1\tilde{R}^{(1)}_2h_3, l_3\tilde{u}^{(1)}_2)\sigma^{-1}(R^{(2)}{}_2, \tilde{R}^{(1)}_3h_4)\\
&\otimes \sigma(\tilde{R}^{(2)}_1h_5, l_4\tilde{u}^{(2)}{}_1)(\tilde{R}^{(2)}_2h_6l_5\tilde{u}^{(2)}_2)\sigma^{-1}(\tilde{R}^{(2)}_3h_7, l_6\tilde{u}^{(2)}_3)\\
\overset{(h)}{=}&\sum \sigma(h_1, l_1)(R^{(1)}\tilde{R}^{(1)}_1h_2l_2\tilde{u}^{(1)}{}_1)\sigma^{-1}(R^{(2)}\tilde{R}^{(1)}_2h_3, l_3\tilde{u}^{(1)}_2)\\
&\otimes \sigma(\tilde{R}^{(2)}_1h_4, l_4\tilde{u}^{(2)}{}_1)(\tilde{R}^{(2)}_2h_5l_5\tilde{u}^{(2)}_2)\sigma^{-1}(\tilde{R}^{(2)}_3h_6, l_6\tilde{u}^{(2)}_3)\\
\overset{(1')}{=}&\sum \sigma(h_1, l_1)(\tilde{R}^{(1)}h_2l_2\tilde{u}^{(1)}{}_1)\sigma^{-1}(R^{(1)}\tilde{R}^{(2)}_1h_3, l_3\tilde{u}^{(1)}_2)\\
&\otimes \sigma(R^{(2)}_1\tilde{R}^{(2)}_2h_4, l_4\widetilde{u^{(2)}}{}_1)(R^{(2)}_2\tilde{R}^{(2)}_3h_5l_5\tilde{u}^{(2)}_2)\sigma^{-1}(R^{(2)}_3\tilde{R}^{(2)}_4h_6, l_6\tilde{u}^{(2)}_3)\\
\overset{(iv)}{=}&\sum \sigma(h_1, l_1)(\tilde{R}^{(1)}h_2l_2\tilde{u}^{(1)}{}_1)\sigma^{-1}(\tilde{R}^{(2)}_1h_3, l_3\tilde{u}^{(1)}_2)\\
&\otimes \sigma(\tilde{R}^{(2)}_2h_4, l_4\tilde{u}^{(2)}{}_1)(\tilde{R}^{(2)}_3h_5l_5\tilde{u}^{(2)}_2)\sigma^{-1}(\tilde{R}^{(2)}_4h_6, l_6\tilde{u}^{(2)}_3)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(p)}{=} \sum \sigma(h_1, l_1)(\tilde{R}^{(1)} h_2 l_2 \tilde{u}^{(1)}{}_1 u^{(1)}) \\
&\quad \sigma^{-1}(\tilde{R}_2^{(2)} h_4, l_4 \tilde{u}_3^{(1)}) \sigma^{-1}(\tilde{R}_1^{(1)} h_3 l_3 \tilde{u}^{(1)}{}_2, u^{(2)}) \\
&\quad \otimes \sigma(\tilde{R}_2^{(2)} h_5, l_5 \tilde{u}^{(2)}{}_1)(\tilde{R}_3^{(2)} h_6 l_6 \tilde{u}_2^{(2)}) \sigma^{-1}(\tilde{R}_4^{(2)} h_7, l_7 \tilde{u}_3^{(2)}) \\
&\overset{(3)}{=} \sum \sigma(h_1, l_1)(\tilde{R}^{(1)} h_2 l_2 \tilde{u}^{(1)}{}_1 u^{(1)}) \\
&\quad \sigma^{-1}(\tilde{R}_1^{(2)} h_3, l_3 \tilde{u}_2^{(1)} u^{(2)}{}_1) \sigma^{-1}(l_4 \tilde{u}_3^{(1)}, u^{(2)}{}_2) \\
&\quad \otimes \sigma(\tilde{R}_2^{(2)} h_4, l_5 \tilde{u}^{(2)}{}_1)(\tilde{R}_3^{(2)} h_6 l_6 \tilde{u}_2^{(2)}) \sigma^{-1}(\tilde{R}_4^{(2)} h_7, l_7 \tilde{u}_3^{(2)}) \\
&\overset{(l)}{=} \sum \sigma(h_1, l_1)(\tilde{R}^{(1)} h_2 l_2 \tilde{u}^{(1)}{}_1 u^{(1)}) \sigma^{-1}(\tilde{R}_1^{(2)} h_3, l_3 \tilde{u}_2^{(1)} u^{(2)}) \\
&\quad \otimes \sigma(\tilde{R}_2^{(2)} h_4, l_4 \tilde{u}^{(2)}{}_1)(\tilde{R}_3^{(2)} h_5 l_5 \tilde{u}_2^{(2)}) \sigma^{-1}(\tilde{R}_4^{(2)} h_6, l_6 \tilde{u}_3^{(2)}) \\
&\overset{(3')}{=} \sum \sigma(h_1, l_1)(\tilde{R}^{(1)} h_2 l_2 \tilde{u}^{(1)}) \sigma^{-1}(\tilde{R}_1^{(2)} h_3, l_3 \tilde{u}_1^{(2)} u^{(1)}) \\
&\quad \otimes \sigma(\tilde{R}_2^{(2)} h_4, l_4 \tilde{u}^{(2)}{}_2 u_1^{(2)})(\tilde{R}_3^{(2)} h_5 l_5 \tilde{u}_3^{(2)} u_3^{(2)}) \sigma^{-1}(\tilde{R}_4^{(2)} h_6, l_6 \tilde{u}_4^{(2)} u_3^{(2)}) \\
&\overset{(r)}{=} \sum \sigma(h_1, l_1)(\tilde{R}^{(1)} h_2 l_2 \tilde{u}^{(1)}) \sigma^{-1}(\tilde{R}_1^{(2)} h_3, l_3 \tilde{u}_1^{(2)}) \\
&\quad \otimes \sigma(\tilde{R}_2^{(2)} h_4, l_4 \tilde{u}^{(2)}{}_2)(\tilde{R}_3^{(2)} h_5 l_5 \tilde{u}_3^{(2)}) \sigma^{-1}(\tilde{R}_4^{(2)} h_6, l_6 \tilde{u}_4^{(2)}) \\
&= \sum \sigma(h_1, l_1)(\tilde{R}^{(1)} h_2 l_2 \tilde{u}^{(1)}) \otimes \tilde{R}_1^{(2)} h_3 l_3 \tilde{u}^{(2)}{}_1 \sigma^{-1}(\tilde{R}_2^{(2)} h_4, l_4 \tilde{u}_2^{(2)}) \\
&\overset{(iv)}{=} \sum \sigma(h_1, l_1)(\tilde{R}^{(1)} h_2 l_2 \tilde{u}^{(1)}) \otimes (R^{(1)} \tilde{R}_1^{(2)} h_3 l_3 \tilde{u}^{(2)}{}_1) \\
&\quad \sigma^{-1}(R^{(2)} \tilde{R}_2^{(2)} h_4, l_4 \tilde{u}_2^{(2)}) \\
&\overset{(1')}{=} \sum \sigma(h_1, l_1)(R^{(1)} \tilde{R}_1^{(1)} h_2 l_2 \tilde{u}^{(1)}) \otimes (R^{(2)} \tilde{R}_2^{(1)} h_3 l_3 \tilde{u}^{(2)}{}_1) \\
&\quad \sigma^{-1}(\tilde{R}^{(2)} h_4, l_4 \tilde{u}_2^{(2)}) \\
&\overset{(iv)}{=} \sum \sigma(h_1, l_1)(R^{(1)} h_2 l_2 \tilde{u}^{(1)}) \otimes (R^{(2)} h_3 l_3 \tilde{u}^{(2)}{}_1) \sigma^{-1}(h_4, l_4 \tilde{u}_2^{(2)}) \\
&\overset{(s)}{=} \sum \sigma(h_1, l_1)(R^{(1)} h_2 l_2 \tilde{u}^{(1)}) \otimes (R^{(2)} h_3 l_3 \tilde{u}^{(2)}{}_1 u^{(1)}) \sigma^{-1}(h_4, l_4 \tilde{u}_2^{(2)} u^{(2)}) \\
&\overset{(3')}{=} \sum \sigma(h_1, l_1)(R^{(1)} h_2 l_2 \tilde{u}_1^{(1)} u^{(1)}) \otimes (R^{(2)} h_3 l_3 \tilde{u}^{(1)}{}_2 u^{(2)}) \sigma^{-1}(h_4, l_4 \tilde{u}^{(2)}) \\
&\overset{(s)}{=} \sum \sigma(h_1, l_1)(R^{(1)} h_2 l_2 u^{(1)}) \otimes (R^{(2)} h_3 l_3 u^{(2)}) \sigma^{-1}(h_4, l_4) \\
&\overset{(A)+(B)}{=} \tilde{\Delta}(h \odot l). \qquad \square
\end{aligned}
$$

Now we will give the Hopf structure of $H^{\sigma-R}$, and we will always assume that the following two conditions hold: $\forall\, h, l \in H$

(t) $\sum \sigma^{-1}(h, lS(U^{(1)}{}_1))S(U^{(1)}{}_2) \otimes U^{(2)} = \sum \sigma^{-1}(h, l)S(U^{(1)}) \otimes U^{(2)}$,

(u) $\sum \sigma^{-1}(S(U^{(2)})hS(R^{(2)}), lS(r^{(2)}))U^{(1)} \otimes R^{(1)} \otimes r^{(1)} = \sigma^{-1}(h, l)$.

Theorem 1.5. *Let H be a $\sigma - R$ compatible Hopf algebra. Suppose that conditions* (t), (u) *hold. Then $(H, \odot, \widetilde{\Delta})$ is a Hopf algebra with the antipode:*

$$S^{\sigma-R}(h) = \sum \sigma(h_1, S(h_2))R^{(1)}S(U^{(1)}h_3R^{(2)})U^{(2)}\sigma^{-1}(S(h_4), h_5).$$

PROOF: By Theorem 1.4, $H^{\sigma-R}$ is a bialgebra. It remains only to verify that $S^{\sigma-R}$ is the antipode of $H^{\sigma-R}$. For all $h \in H$

$$\begin{aligned}
&(I \star S^{\sigma-R})(h)\\
&\overset{(B)}{=} \sum (\tilde{R}^{(1)}h_1\tilde{U}^{(1)}) \odot S^{\sigma-R}(\tilde{R}^{(2)}h_2\tilde{U}^{(2)})\\
&\overset{(A)}{=} \sum \sigma(\tilde{R}^{(1)}{}_1h_1\tilde{U}^{(1)}{}_1, S^{\sigma-R}(\tilde{R}^{(2)}h_4\tilde{U}^{(2)})_1)\\
&\quad [(\tilde{R}^{(1)}{}_2h_2\tilde{U}^{(1)}{}_2)S^{\sigma-R}(\tilde{R}^{(2)}h_4\tilde{U}^{(2)})_2]\\
&\quad \sigma^{-1}(\tilde{R}^{(1)}{}_3h_3\tilde{U}^{(1)}{}_3, S^{\sigma-R}(\tilde{R}^{(2)}h_4\tilde{U}^{(2)})_3)\\
&= \sum \sigma(\tilde{R}^{(1)}{}_1h_1\tilde{U}^{(1)}{}_1, (R^{(1)}S(U^{(1)}\tilde{R}^{(2)}_3h_6\tilde{U}^{(2)}_3R^{(2)})U^{(2)})_1)\\
&\quad \sigma(\tilde{R}^{(2)}{}_1h_4\tilde{U}^{(2)}{}_1, S(\tilde{R}^{(2)}_2h_5\tilde{U}^{(2)}_2))\\
&\quad [\tilde{R}^{(1)}{}_2h_2\tilde{U}^{(1)}{}_2(R^{(1)}S(U^{(1)}\tilde{R}^{(2)}_3h_6\tilde{U}^{(2)}{}_3R^{(2)})U^{(2)})_2]\\
&\quad \sigma^{-1}(\tilde{R}^{(1)}{}_3h_3\tilde{U}^{(1)}{}_3, (R^{(1)}S(U^{(1)}\tilde{R}^{(2)}{}_3h_6\tilde{U}^{(2)}{}_3R^{(2)})U^{(2)})_3)\\
&\quad \sigma^{-1}(S(\tilde{R}^{(2)}{}_4h_7\tilde{U}^{(2)}_4), \tilde{R}^{(2)}{}_5h_8\tilde{U}^{(2)}{}_5)\\
&\overset{(i)+(2')}{=} \sum \sigma(r^{(1)}\tilde{R}^{(1)}{}_1h_1\tilde{U}^{(1)}{}_1, R^{(1)}{}_1S(U^{(1)}\tilde{R}^{(2)}_3h_6\tilde{U}^{(2)}_3R^{(2)})_1U^{(2)}{}_1)\\
&\quad \sigma(\tilde{R}^{(2)}{}_1h_4\tilde{U}^{(2)}{}_1, S(\tilde{R}^{(2)}_2h_5\tilde{U}^{(2)}_2))\\
&\quad [r^{(2)}{}_1\tilde{R}^{(1)}{}_2h_2\tilde{U}^{(1)}{}_2R^{(1)}{}_2S(U^{(1)}\tilde{R}^{(2)}_3h_6\tilde{U}^{(2)}_3R^{(2)})_2U^{(2)}{}_2]\\
&\quad \sigma^{-1}(r^{(2)}{}_2\tilde{R}^{(1)}{}_3h_3\tilde{U}^{(1)}{}_3, R^{(1)}{}_3S(U^{(1)}\tilde{R}^{(2)}{}_3h_6\\
&\quad \tilde{U}^{(2)}{}_3R^{(2)})_3U^{(2)}{}_3)\sigma^{-1}(S(\tilde{R}^{(2)}{}_4h_7\tilde{U}^{(2)}{}_4), \tilde{R}^{(2)}{}_5h_8\tilde{U}^{(2)}{}_5)\\
&\overset{(1')}{=} \sum \sigma(\tilde{R}^{(1)}h_1\tilde{U}^{(1)}{}_1, R^{(1)}{}_1S(U^{(1)}r^{(2)}{}_3\tilde{R}^{(2)}{}_4h_6\tilde{U}^{(2)}{}_3R^{(2)})_1U^{(2)}_1)\\
&\quad \sigma(r^{(2)}{}_1\tilde{R}^{(1)}{}_2h_4\tilde{U}^{(2)}{}_1, S(r^{(2)}{}_2\tilde{R}^{(2)}{}_3h_5\tilde{U}^{(2)}{}_2))\\
&\quad [r^{(1)}{}_1\tilde{R}^{(2)}{}_1h_2\tilde{U}^{(1)}{}_2R^{(1)}{}_2S(U^{(1)}r^{(2)}{}_3\tilde{R}^{(2)}{}_4h_6\tilde{U}^{(2)}{}_3R^{(2)})_2U^{(2)}{}_2]\\
&\quad \sigma^{-1}(r^{(1)}{}_2\tilde{R}^{(2)}{}_2h_3\tilde{U}^{(1)}{}_3, R^{(1)}{}_3\\
&\quad S(U^{(1)}r^{(2)}{}_3\tilde{R}^{(2)}{}_4h_6\tilde{U}^{(2)}{}_3R^{(2)})_3U^{(2)}{}_3)\\
&\quad \sigma^{-1}(S(r^{(2)}{}_4\tilde{R}^{(2)}{}_5h_7\tilde{U}^{(2)}{}_4), r^{(2)}{}_5\tilde{R}^{(2)}{}_6h_8\tilde{U}^{(2)}{}_5)\\
&\overset{(2')+(i)}{=} \sum \sigma(h_1\tilde{U}^{(1)}{}_1, R^{(1)}{}_1S(U^{(1)}r^{(2)}{}_3h_6\tilde{U}^{(2)}{}_3R^{(2)})_1U^{(2)}_1)
\end{aligned}$$

$$\sigma(r^{(2)}{}_1h_4\tilde{U}^{(2)}{}_1, S(r^{(2)}{}_2h_5\tilde{U}^{(2)}{}_2))$$
$$[r^{(1)}{}_1h_2\tilde{U}^{(1)}{}_2R^{(1)}{}_2S(U^{(1)}r^{(2)}{}_3h_6\tilde{U}^{(2)}{}_3R^{(2)})_2U^{(2)}{}_2]$$
$$\sigma^{-1}(r^{(1)}{}_2h_3\tilde{U}^{(1)}{}_3, R^{(1)}{}_3S(U^{(1)}r^{(2)}{}_3h_6$$
$$\tilde{U}^{(2)}{}_3R^{(2)})_3U^{(2)}{}_3)\sigma^{-1}(S(r^{(2)}{}_4h_7\tilde{U}^{(2)}{}_4), r^{(2)}{}_5h_8\tilde{U}^{(2)}{}_5)$$
$$\overset{(i)+(2')}{=} \sum \sigma(h_1\tilde{U}^{(1)}{}_1, \tilde{r}^{(1)}R^{(1)}{}_1S(U^{(1)}r^{(2)}{}_3h_6\tilde{U}^{(2)}{}_3R^{(2)})_1U_1^{(2)})$$
$$\sigma(r^{(2)}{}_1h_4\tilde{U}^{(2)}{}_1, S(r^{(2)}{}_2h_5\tilde{U}^{(2)}{}_2))$$
$$[r^{(1)}{}_1h_2\tilde{U}^{(1)}{}_2\tilde{r}^{(2)}{}_1R^{(1)}{}_2S(U^{(1)}r^{(2)}{}_3h_6\tilde{U}^{(2)}{}_3R^{(2)})_2U^{(2)}{}_2]$$
$$\sigma^{-1}(r^{(1)}{}_2h_3\tilde{U}^{(1)}{}_3, \tilde{r}^{(2)}{}_2R^{(1)}{}_3S(U^{(1)}r^{(2)}{}_3h_6\tilde{U}^{(2)}{}_3R^{(2)})_3U^{(2)}{}_3)$$
$$\sigma^{-1}(S(r^{(2)}{}_4h_7\tilde{U}^{(2)}{}_4), r^{(2)}{}_5h_8\tilde{U}^{(2)}{}_5)$$
$$\overset{(1')}{=} \sum \sigma(h_1\tilde{U}^{(1)}{}_1, R^{(1)}S(U^{(1)}r^{(2)}{}_3h_6\tilde{U}^{(2)}{}_3\tilde{r}^{(2)}R^{(2)}{}_3)_1U_1^{(2)})$$
$$\sigma(r^{(2)}{}_1h_4\tilde{U}^{(2)}{}_1, S(r^{(2)}{}_2h_5\tilde{U}^{(2)}{}_2))$$
$$[r^{(1)}{}_1h_2\tilde{U}^{(1)}{}_2\tilde{r}^{(1)}{}_1R^{(2)}{}_1S(U^{(1)}r^{(2)}{}_3h_6\tilde{U}^{(2)}{}_3\tilde{r}^{(2)}R^{(2)}{}_3)_2U^{(2)}{}_2]$$
$$\sigma^{-1}(r^{(1)}{}_2h_3\tilde{U}^{(1)}{}_3, \tilde{r}^{(1)}{}_2R^{(2)}{}_2$$
$$S(U^{(1)}r^{(2)}{}_3h_6\tilde{U}^{(2)}{}_3\tilde{r}^{(2)}R^{(2)}{}_3)_3U^{(2)}{}_3)$$
$$\sigma^{-1}(S(r^{(2)}{}_4h_7\tilde{U}^{(2)}{}_4), r^{(2)}{}_5h_8\tilde{U}^{(2)}{}_5)$$
$$\overset{(2')+(i)}{=} \sum \sigma(h_1\tilde{U}^{(1)}{}_1, S(U^{(1)}r^{(2)}{}_3h_6\tilde{U}^{(2)}{}_3\tilde{r}^{(2)})_1U_1^{(2)})\sigma(r^{(2)}{}_1h_4\tilde{U}^{(2)}{}_1,$$
$$S(r^{(2)}{}_2h_5\tilde{U}^{(2)}{}_2))$$
$$[r^{(1)}{}_1h_2\tilde{U}^{(1)}{}_2\tilde{r}^{(1)}{}_1S(U^{(1)}r^{(2)}{}_3h_6\tilde{U}^{(2)}{}_3\tilde{r}^{(2)})_2U^{(2)}{}_2]$$
$$\sigma^{-1}(r^{(1)}{}_2h_3\tilde{U}^{(1)}{}_3, \tilde{r}^{(1)}{}_2S(U^{(1)}r^{(2)}{}_3h_6\tilde{U}^{(2)}{}_3\tilde{r}^{(2)})_3U^{(2)}{}_3)$$
$$\sigma^{-1}(S(r^{(2)}{}_4h_7\tilde{U}^{(2)}{}_4), r^{(2)}{}_5h_8\tilde{U}^{(2)}{}_5)$$
$$\overset{(d)+(4')}{=} \sum \sigma(h_1\tilde{U}^{(1)}{}_1u^{(1)}, S(U^{(1)}r^{(2)}{}_3h_6\tilde{U}^{(2)}{}_3\tilde{r}^{(2)})_1U_1^{(2)})$$
$$\sigma(r^{(2)}{}_1h_4\tilde{U}^{(2)}{}_1, S(r^{(2)}{}_2h_5\tilde{U}^{(2)}{}_2))$$
$$[r^{(1)}{}_1h_2\tilde{U}^{(1)}{}_2u^{(2)}{}_1\tilde{r}^{(1)}{}_1S(U^{(1)}r^{(2)}{}_3h_6\tilde{U}^{(2)}{}_3\tilde{r}^{(2)})_2U^{(2)}{}_2]$$
$$\sigma^{-1}(r^{(1)}{}_2h_3\tilde{U}^{(1)}{}_3u^{(2)}{}_2, \tilde{r}^{(1)}{}_2S(U^{(1)}r^{(2)}{}_3h_6\tilde{U}^{(2)}{}_3\tilde{r}^{(2)})_3U^{(2)}{}_3)$$
$$\sigma^{-1}(S(r^{(2)}{}_4h_7\tilde{U}^{(2)}{}_4), r^{(2)}{}_5h_8\tilde{U}^{(2)}{}_5)$$
$$\overset{(3')}{=} \sum \sigma(h_1\tilde{U}^{(1)}, S(U^{(1)}r^{(2)}{}_3h_6\tilde{U}^{(2)}{}_4u^{(2)}{}_3\tilde{r}^{(2)})_1U_1^{(2)})$$
$$\sigma(r^{(2)}{}_1h_4\tilde{U}^{(2)}{}_2u^{(2)}{}_1, S(r^{(2)}{}_2h_5\tilde{U}^{(2)}{}_3u^{(2)}{}_2))$$
$$[r^{(1)}{}_1h_2\tilde{U}^{(2)}{}_1u^{(1)}{}_1\tilde{r}^{(1)}{}_1S(U^{(1)}r^{(2)}{}_3h_6\tilde{U}^{(2)}{}_3u^{(2)}{}_3\tilde{r}^{(2)})_02U^{(2)}{}_2]$$

$$\sigma^{-1}(r^{(1)}{}_2h_3\tilde{U}^{(2)}{}_2u^{(1)}{}_2, \tilde{r}^{(1)}{}_2S(U^{(1)}r^{(2)}{}_3h_6\tilde{U}^{(2)}{}_4u^{(2)}{}_3\tilde{r}^{(2)})_3U^{(2)}{}_3)$$
$$\sigma^{-1}(S(r^{(2)}{}_4h_7\tilde{U}^{(2)}{}_5u^{(2)}{}_4), r^{(2)}{}_5h_8\tilde{U}^{(2)}{}_6u^{(2)}{}_5)$$

$$\overset{(d)+(4')}{=}\sum\sigma(h_1, S(U^{(1)}r^{(2)}{}_3h_6u^{(2)}{}_3\tilde{r}^{(2)})_1U_1^{(2)}\tilde{u}^{(1)})$$
$$\sigma(r^{(2)}{}_1h_4u^{(2)}{}_1, S(r^{(2)}{}_2h_5u^{(2)}{}_2))$$
$$[r^{(1)}{}_1h_2u^{(1)}{}_1\tilde{r}^{(1)}{}_1S(U^{(1)}r^{(2)}{}_3h_6u^{(2)}{}_3\tilde{r}^{(2)})_2U^{(2)}{}_2\tilde{u}^{(2)}{}_1]$$
$$\sigma^{-1}(r^{(1)}{}_2h_3u^{(1)}{}_2, \tilde{r}^{(1)}{}_2S(U^{(1)}r^{(2)}{}_3h_6u^{(2)}{}_3\tilde{r}^{(2)})_3U^{(2)}{}_3\tilde{u}^{(2)}{}_2)$$
$$\sigma^{-1}(S(r^{(2)}{}_4h_7u^{(2)}{}_4), r^{(2)}{}_5h_8u^{(2)}{}_5)$$

$$\overset{(3')}{=}\sum\sigma(h_1, S(U^{(1)}{}_1\tilde{u}^{(1)}r^{(2)}{}_3h_6u^{(2)}{}_3\tilde{r}^{(2)})_1U_2^{(1)}\tilde{u}^{(2)})$$
$$\sigma(r^{(2)}{}_1h_4u^{(2)}{}_1, S(r^{(2)}{}_2h_5u^{(2)}{}_2))$$
$$[r^{(1)}{}_1h_2u^{(1)}{}_1\tilde{r}^{(1)}{}_1S(U^{(1)}{}_{(1)}\tilde{u}^{(1)}r^{(2)}{}_3h_6u^{(2)}{}_3\tilde{r}^{(2)})_2U^{(2)}{}_1]$$
$$\sigma^{-1}(r^{(1)}{}_2h_3u^{(1)}{}_2, \tilde{r}^{(1)}{}_2S(U^{(1)}{}_1\tilde{u}^{(1)}r^{(2)}{}_3h_6u^{(2)}{}_3\tilde{r}^{(2)})_3U^{(2)}{}_2)$$
$$\sigma^{-1}(S(r^{(2)}{}_4h_7u^{(2)}{}_4), r^{(2)}{}_5h_8u^{(2)}{}_5)$$

$$\overset{(e)+(4')}{=}\sum\sigma(h_1, S(r^{(2)}{}_3h_6u^{(2)}{}_3\tilde{r}^{(2)})_1U_2^{(1)}\tilde{u}^{(2)})$$
$$\sigma(r^{(2)}{}_1h_4u^{(2)}{}_1, S(r^{(2)}{}_2h_5u^{(2)}{}_2))$$
$$[r^{(1)}{}_1h_2u^{(1)}{}_1\tilde{r}^{(1)}{}_1S(r^{(2)}{}_3h_6u^{(2)}{}_3\tilde{r}^{(2)})_2S(U^{(1)}{}_2)U^{(2)}{}_1]$$
$$\sigma^{-1}(r^{(1)}{}_2h_3u^{(1)}{}_2, \tilde{r}^{(1)}{}_2S(r^{(2)}{}_3h_6u^{(2)}{}_3\tilde{r}^{(2)})_3S(U^{(1)}{}_1)U^{(2)}{}_2)$$
$$\sigma^{-1}(S(r^{(2)}{}_4h_7u^{(2)}{}_4), r^{(2)}{}_5h_8u^{(2)}{}_5)$$

$$\overset{(s)}{=}\sum\sigma(h_1, S(r^{(2)}{}_3h_6u^{(2)}{}_3\tilde{r}^{(2)})_1U_2^{(1)}\tilde{u}^{(2)})$$
$$\sigma(r^{(2)}{}_1h_4u^{(2)}{}_1, S(r^{(2)}{}_2h_5u^{(2)}{}_2))$$
$$[r^{(1)}{}_1h_2u^{(1)}{}_1\tilde{r}^{(1)}{}_1S(r^{(2)}{}_3h_6u^{(2)}{}_3\tilde{r}^{(2)})_2S(U^{(1)}{}_2)U^{(2)}{}_1\tilde{u}^{(1)}]$$
$$\sigma^{-1}(r^{(1)}{}_2h_3u^{(1)}{}_2, \tilde{r}^{(1)}{}_2S(r^{(2)}{}_3h_6u^{(2)}{}_3\tilde{r}^{(2)})_3S(U^{(1)}{}_1)U^{(2)}{}_2\tilde{u}^{(2)})$$
$$\sigma^{-1}(S(r^{(2)}{}_4h_7u^{(2)}{}_4), r^{(2)}{}_5h_8u^{(2)}{}_5)$$

$$\overset{(3')}{=}\sum\sigma(h_1, S(r^{(2)}{}_3h_6u^{(2)}{}_3\tilde{r}^{(2)})_1U_2^{(1)}\tilde{u}^{(2)})$$
$$\sigma(r^{(2)}{}_1h_4u^{(2)}{}_1, S(r^{(2)}{}_2h_5u^{(2)}{}_2))$$
$$[r^{(1)}{}_1h_2u^{(1)}{}_1\tilde{r}^{(1)}{}_1S(r^{(2)}{}_3h_6u^{(2)}{}_3\tilde{r}^{(2)})_2S(U^{(1)}{}_2\tilde{u}^{(1)}{}_2)U^{(1)}{}_3\tilde{u}^{(2)}]$$
$$\sigma^{-1}(r^{(1)}{}_2h_3u^{(1)}{}_2, \tilde{r}^{(1)}{}_2S(r^{(2)}{}_3h_6u^{(2)}{}_3\tilde{r}^{(2)})_3S(U^{(1)}{}_1\tilde{u}^{(1)}{}_1)U^{(2)})$$
$$\sigma^{-1}(S(r^{(2)}{}_4h_7u^{(2)}{}_4), r^{(2)}{}_5h_8u^{(2)}{}_5)$$

$$\begin{aligned}
&\overset{(s)}{=} \sum \sigma(h_1, S(r^{(2)}{}_3 h_6 u^{(2)}{}_3 \tilde{r}^{(2)})_1 U_2^{(1)} \tilde{u}^{(2)}) \\
&\quad \sigma(r^{(2)}{}_1 h_4 u^{(2)}{}_1, S(r^{(2)}{}_2 h_5 u^{(2)}{}_2)) \\
&\quad [r^{(1)}{}_1 h_2 u^{(1)}{}_1 \tilde{r}^{(1)}{}_1 S(r^{(2)}{}_3 h_6 u^{(2)}{}_3 \tilde{r}^{(2)})_2 S(\tilde{u}^{(1)}{}_2) \tilde{u}^{(2)}] \\
&\quad \sigma^{-1}(r^{(1)}{}_2 h_3 u^{(1)}{}_2, \tilde{r}^{(1)}{}_2 S(r^{(2)}{}_3 h_6 u^{(2)}{}_3 \tilde{r}^{(2)})_3 S(\tilde{u}^{(1)}{}_1)) \\
&\quad \sigma^{-1}(S(r^{(2)}{}_4 h_7 u^{(2)}{}_4), r^{(2)}{}_5 h_8 u^{(2)}{}_5) \\
&\overset{(iv)}{=} \sum \sigma(h_1, S(R^{(1)}{}_3 r^{(2)}{}_3 h_6 u^{(2)}{}_3 \tilde{r}^{(2)})_1 U_2^{(1)} \tilde{u}^{(2)}) \\
&\quad \sigma(R^{(1)}{}_1 r^{(2)}{}_1 h_4 u^{(2)}{}_1, S(R^{(1)}{}_2 r^{(2)}{}_2 h_5 u^{(2)}{}_2)) \\
&\quad [r^{(1)}{}_1 h_2 u^{(1)}{}_1 \tilde{r}^{(1)}{}_1 S(R^{(1)}{}_3 r^{(2)}{}_3 h_6 u^{(2)}{}_3 \tilde{r}^{(2)})_2 S(\tilde{u}^{(1)}{}_2) \tilde{u}^{(2)}] \\
&\quad \sigma^{-1}(r^{(1)}{}_2 h_3 u^{(1)}{}_2, \tilde{r}^{(1)}{}_2 S(R^{(1)}{}_3 r^{(2)}{}_3 h_6 u^{(2)}{}_3 \tilde{r}^{(2)})_3 S(\tilde{u}^{(1)}{}_1)) \\
&\quad \sigma^{-1}(S(R^{(1)}{}_4 r^{(2)}{}_4 h_7 u^{(2)}{}_4), R^{(2)} r^{(2)}{}_5 h_8 u^{(2)}{}_5) \\
&\overset{(1')}{=} \sum \sigma(h_1, S(R^{(2)}{}_3 r^{(1)}{}_5 h_6 u^{(2)}{}_3 \tilde{r}^{(2)})_1 U_2^{(1)} \tilde{u}^{(2)}) \\
&\quad \sigma(R^{(2)}{}_1 r^{(1)}{}_3 h_4 u^{(2)}{}_1, S(R^{(2)}{}_2 r^{(1)}{}_4 h_5 u^{(2)}{}_2)) \\
&\quad [R^{(1)}{}_1 r^{(1)}{}_1 h_2 u^{(1)}{}_1 \tilde{r}^{(1)}{}_1 S(R^{(2)}{}_3 r^{(1)}{}_5 h_6 u^{(2)}{}_3 \tilde{r}^{(2)})_2 S(\tilde{u}^{(1)}{}_2) \tilde{u}^{(2)}] \\
&\quad \sigma^{-1}(R^{(1)}{}_2 r^{(1)}{}_2 h_3 u^{(1)}{}_2, \tilde{r}^{(1)}{}_2 S(R^{(2)}{}_3 r^{(1)}{}_5 h_6 u^{(2)}{}_3 \tilde{r}^{(2)})_3 S(\tilde{u}^{(1)}{}_1)) \\
&\quad \sigma^{-1}(S(R^{(2)}{}_4 r^{(1)}{}_6 h_7 u^{(2)}{}_4), r^{(2)} h_8 u^{(2)}{}_5) \\
&\overset{(iv)+(s)}{=} \sum \sigma(h_1, S(R^{(2)}{}_3 h_6 u^{(2)}{}_3 \tilde{r}^{(2)})_1 U^{(1)}{}_2 \tilde{u}^{(2)}) \\
&\quad \sigma(R^{(2)}{}_1 h_4 u^{(2)}{}_1 U^{(1)}{}_1, S(R^{(2)}{}_2 h_5 u^{(2)}{}_2 U^{(1)}{}_2)) \\
&\quad [R^{(1)}{}_1 h_2 u^{(1)}{}_1 \tilde{r}^{(1)}{}_1 S(R^{(2)}{}_3 h_6 u^{(2)}{}_3 U^{(1)}{}_3 \tilde{r}^{(2)})_2 S(\tilde{u}^{(1)}{}_2) \tilde{u}^{(2)}] \\
&\quad \sigma^{-1}(R^{(1)}{}_2 h_3 u^{(1)}{}_2, \tilde{r}^{(1)}{}_2 S(R^{(2)}{}_3 h_6 u^{(2)}{}_3 \tilde{r}^{(2)})_3 S(\tilde{u}^{(1)}{}_1)) \\
&\quad \sigma^{-1}(S(R^{(2)}{}_4 h_7 u^{(2)}{}_4 U^{(1)}{}_4), h_8 u^{(2)}{}_5 U^{(2)}) \\
&\overset{(3')}{=} \sum \sigma(h_1, S(R^{(2)}{}_3 h_6 u^{(2)}{}_3 \tilde{r}_1^{(2)})) \\
&\quad \sigma(R^{(2)}{}_1 h_4 u^{(1)}{}_3 U^{(2)}{}_1, S(R^{(2)}{}_2 h_5 u^{(1)}{}_4 U^{(2)}{}_2)) \\
&\quad [R^{(1)}{}_1 h_2 u^{(1)}{}_1 U^{(1)}{}_1 \tilde{r}^{(1)}{}_1 S(R^{(2)}{}_3 h_6 u^{(1)}{}_5 U^{(2)}{}_3 \tilde{r}^{(2)})_2 S(\tilde{u}^{(1)}{}_2) \tilde{u}^{(2)}] \\
&\quad \sigma^{-1}(R^{(1)}{}_2 h_3 u^{(1)}{}_2 U^{(1)}{}_2, \tilde{r}^{(1)}{}_2 S(R^{(2)}{}_3 h_6 u^{(1)}{}_5 U^{(2)}{}_3 \tilde{r}^{(2)})_3 S(\tilde{u}^{(1)}{}_1)) \\
&\quad \sigma^{-1}(S(R^{(2)}{}_4 h_7 u^{(1)}{}_6 U^{(2)}{}_4), h_8 u^{(2)}) \\
&\overset{(s)+(iv)}{=} \sum \sigma(h_1, S(R^{(2)}{}_5 h_8 U^{(2)}{}_5 \tilde{r}^{(2)}{}_3)) \\
&\quad \sigma(R^{(2)}{}_1 h_4 U^{(2)}{}_1, S(R^{(2)}{}_2 h_5 U^{(2)}{}_2))
\end{aligned}$$

$$
\begin{aligned}
&[R^{(1)}{}_1h_2U^{(1)}{}_1\tilde{R}^{(1)}\tilde{r}^{(1)}{}_1S(R^{(2)}{}_4h_7U^{(2)}{}_4\tilde{r}^{(2)}{}_2)S(\tilde{u}^{(1)}{}_2)\tilde{u}^{(2)}]\\
&\sigma^{-1}(R^{(1)}{}_2h_3U^{(1)}{}_2,\tilde{R}^{(2)}\tilde{r}^{(1)}{}_2S(R^{(2)}{}_3h_6U^{(2)}{}_3\tilde{r}^{(2)}{}_1)S(\tilde{u}^{(1)}{}_1))\\
&\sigma^{-1}(S(R^{(2)}{}_4h_9U^{(2)}{}_4),h_{10})\\
\overset{(1')}{=}&\sum\sigma(h_1,S(R^{(2)}{}_5h_8U^{(2)}{}_5\tilde{R}^{(2)}{}_3\tilde{r}^{(2)}_4))\\
&\sigma(R^{(2)}{}_1h_4U^{(2)}{}_1,S(R^{(2)}{}_2h_5U^{(2)}{}_2))\\
&[R^{(1)}{}_1h_2U^{(1)}{}_1\tilde{r}^{(1)}S(R^{(2)}{}_4h_7U^{(2)}{}_4\tilde{R}^{(2)}{}_2\tilde{r}^{(2)}{}_3)S(\tilde{u}^{(1)}{}_2)\tilde{u}^{(2)}]\\
&\sigma^{-1}(R^{(1)}{}_2h_3U^{(1)}{}_2,\tilde{R}^{(1)}\tilde{r}^{(2)}{}_1S(R^{(2)}{}_3h_6U^{(2)}{}_3\tilde{R}^{(2)}{}_1\tilde{r}^{(2)}{}_2)\\
&S(\tilde{u}^{(1)}{}_1))\sigma^{-1}(S(R^{(2)}{}_4h_9U^{(2)}{}_4),h_{10})\\
\overset{(iv)+(s)}{=}&\sum\sigma(h_1,S(R^{(2)}{}_5h_8U^{(2)}{}_5\tilde{r}^{(2)}_2))\sigma(R^{(2)}{}_1h_4U^{(2)}{}_1,S(R^{(2)}{}_2h_5U^{(2)}{}_2))\\
&[r^{(1)}R^{(1)}{}_1h_2U^{(1)}{}_1u^{(1)}\tilde{r}^{(1)}S(R^{(2)}{}_4h_7U^{(2)}{}_4\tilde{r}^{(2)}{}_1)S(\tilde{u}^{(1)}{}_2)\tilde{u}^{(2)}]\\
&\sigma^{-1}(r^{(2)}R^{(1)}{}_2h_3U^{(1)}{}_2u^{(2)},\\
&S(R^{(2)}{}_3h_6U^{(2)}{}_3)S(\tilde{u}^{(1)}{}_1))\sigma^{-1}(S(R^{(2)}{}_6h_9U^{(2)}{}_4),h_{10})\\
\overset{(1')+(3')}{=}&\sum\sigma(h_1,S(r^{(2)}{}_5R^{(2)}{}_6h_8U^{(2)}{}_6u^{(2)}{}_5\tilde{r}^{(2)}_2))\\
&\sigma(r^{(2)}{}_1R^{(2)}{}_2h_4U^{(2)}{}_2u^{(2)}{}_1,S(r^{(2)}{}_2R^{(2)}{}_3h_5U^{(2)}{}_3u^{(2)}{}_2))\\
&[R^{(1)}h_2U^{(1)}\tilde{r}^{(1)}S(r^{(2)}{}_4R^{(2)}{}_5h_7U^{(2)}u^{(2)}\tilde{r}^{(2)}{}_1)S(\tilde{u}^{(1)}{}_2)\tilde{u}^{(2)}]\\
&\sigma^{-1}(r^{(1)}R^{(2)}{}_1h_3U^{(2)}{}_1u^{(1)},S(r^{(2)}{}_3R^{(2)}{}_4h_6U^{(2)}{}_4u^{(2)}{}_3)S(\tilde{u}^{(1)}{}_1))\\
&\sigma^{-1}(S(r^{(2)}{}_6R^{(2)}{}_7h_9U^{(2)}{}_7u^{(2)}{}_6),h_{10})\\
\overset{(r)+(iv)}{=}&\sum\sigma(h_1,S(R^{(2)}{}_6h_8U^{(2)}{}_6\tilde{r}^{(2)}_2))\sigma(R^{(2)}{}_2h_4U^{(2)}{}_2,S(R^{(2)}{}_3h_5U^{(2)}{}_3))\\
&[R^{(1)}h_2U^{(1)}\tilde{r}^{(1)}S(R^{(2)}{}_5h_7U^{(2)}{}_5\tilde{r}^{(2)}{}_1)S(\tilde{u}^{(1)}{}_2)\tilde{u}^{(2)}]\\
&\sigma^{-1}(R^{(2)}{}_1h_3U^{(2)}{}_1,S(R^{(2)}{}_4h_6U^{(2)}{}_4)S(\tilde{u}^{(1)}{}_1))\\
&\sigma^{-1}(S(R^{(2)}{}_7h_9U^{(2)}{}_7),h_{10})\\
\overset{(t)+(u)}{=}&\sum\sigma(h_1,S(R^{(2)}{}_2h_4U^{(2)}{}_2\tilde{u}^{(1)}{}_2\tilde{r}^{(2)}_2))\\
&[R^{(1)}h_2U^{(1)}\tilde{r}^{(1)}S(R^{(2)}{}_1h_3U^{(2)}{}_1\tilde{u}^{(1)}{}_1\tilde{r}^{(2)}{}_1)S(u^{(1)})u^{(2)}]\\
&\sigma^{-1}(S(R^{(2)}{}_3h_5U^{(2)}{}_3\tilde{u}^{(2)}),h_6)\\
\overset{(3')+(u)}{=}&\sum\sigma(h_1,S(R^{(2)}{}_2h_4\tilde{u}^{(2)}{}_2\tilde{r}^{(2)}_2))[R^{(1)}h_2\tilde{u}^{(1)}\tilde{r}^{(1)}S(R^{(2)}{}_1h_3\\
&\tilde{u}^{(2)}{}_1\tilde{r}^{(2)}{}_1)S(u^{(1)})u^{(2)}]\sigma^{-1}(S(h_5)S(R^{(2)}{}_3),h_6)\\
\overset{(u)+(1')}{=}&\sum\sigma(h_1,S(h_2)S(\tilde{R}^{(2)}))[R^{(1)}\tilde{R}^{(1)}{}_1S(R^{(2)}\tilde{R}^{(1)}{}_2)S(u^{(1)})u^{(2)}]
\end{aligned}
$$

$$\sigma^{-1}(S(h_3),h_4)$$

$$\overset{(u)+([1,Thm1.6])}{=}\sum \epsilon(h)[R^{(1)}r^{(1)}{}_1U^{(1)}{}_1S(u^{(1)}R^{(2)}r^{(1)}{}_2U^{(1)}{}_2)u^{(2)}r^{(2)}U^{(2)}]$$

$$\sigma^{-1}(S(h_3),h_4)$$

$$\overset{(1')+(2')}{=}\sum \epsilon(h)$$

Similarly, we can prove that $(S^{\sigma-R}\star I)(h)=\epsilon(h)$. □

Corollary 1.6 ([1, Theorem 1.6]). *Let A be $\sigma-$Hopf algebra. Then A^σ is a Hopf algebra.*

PROOF: Take $R=1\otimes 1$ in Theorem 1.5. Then A is $\sigma-R$ compatible Hopf algebra. This completes the proof. □

Corollary 1.7. *Let H be $R-$Hopf algebra. Then A^R is a Hopf algebra.*

PROOF: Take σ be trivial in Theorem 1.5. Then H is $\sigma-R$ compatible Hopf algebra. This completes the proof. □

2. Braided structure over $H^{\sigma-R}$

The main purpose of this section is to generalize [1, Theorem 1.6(c)].

Theorem 2.1. *Let H be an $\sigma-R-$compatible Hopf algebra, and (H,τ) a braided Hopf algebra. Then $H^{\sigma-R}$ is a braided Hopf algebra with braided structure as follows: For all $h,l\in H$*

$$\tilde{\sigma}(h,l)=\sum \sigma(R^{(1)}(r^{(1)}{}_1h_1u_1^{(1)})U^{(1)},\tilde{R}^{(1)}(\tilde{r}^{(1)}{}_1l_1\tilde{u}^{(1)}{}_1)\tilde{U}^{(1)})$$
$$\tau(\tilde{R}^{(2)}(r^{(1)}{}_2l_2\tilde{u}^{(1)}{}_2)\tilde{U}^{(2)},R^{(2)}(r^{(1)}{}_2h_2u^{(1)}{}_2)U^{(2)})$$
$$\sigma^{-1}(\tilde{r}^{(2)}l_3\tilde{u}^{(2)},r^{(2)}h_3u^{(2)})$$

PROOF: Firstly, the above braided structure of $\tilde{\sigma}(h,l)$ will be simplified as following:

$$\tilde{\sigma}(h,l)\overset{(i)}{=}\sum \sigma((r^{(1)}{}_1h_1u_1^{(1)})U^{(1)},(\tilde{r}^{(1)}{}_1l_1\tilde{u}^{(1)}{}_1)\tilde{U}^{(1)})$$
$$\tau((\tilde{r}^{(1)}{}_2l_2\tilde{u}^{(1)}{}_2)\tilde{U}^{(2)},(r^{(1)}{}_2h_2u^{(1)}{}_2)U^{(2)})\sigma^{-1}(\tilde{r}^{(2)}l_3\tilde{u}^{(2)},r^{(2)}h_3u^{(2)})$$
$$\overset{(d)}{=}\sum \sigma(r^{(1)}{}_1h_1u_1^{(1)},\tilde{r}^{(1)}{}_1l_1\tilde{u}^{(1)}{}_1)\tau(\tilde{r}^{(1)}{}_2l_2\tilde{u}^{(1)}{}_2,r^{(1)}{}_2h_2u^{(1)}{}_2)$$
$$\sigma^{-1}(\tilde{r}^{(2)}l_3\tilde{u}^{(2)},r^{(2)}h_3u^{(2)})$$
$$\overset{(iv)}{=}\sum \sigma(h_1u_1^{(1)},l_1\tilde{u}^{(1)}{}_1)\tau(l_2\tilde{u}^{(1)}{}_2,h_2u^{(1)}{}_2)\sigma^{-1}(l_3\tilde{u}^{(2)},h_3u^{(2)})$$
$$\overset{(s)}{=}\sum \sigma(h_1,l_1)\tau(l_2,h_2)\sigma^{-1}(l_3,h_3)$$

Secondly, $\tilde{\sigma}$ satisfies conditions in [1, Definition 1.1] by straightforward computations.

□

Remark. This theorem shows that R does not affect the braided structure $\tilde{\sigma}$.

Corollary 2.2. *Let H be a $R-$Hopf algebra, and (H, τ) a braided Hopf algebra. Then $(H^{\sigma(\tau)}, \tilde{\sigma})$ is a braided Hopf algebra with braiding:*

$$\tilde{\sigma}(x, y) = \sum \sigma(x_1, y_1)\tau(y_2, x_2)\sigma^{-1}(y_3, x_3)$$

Proof: Let R be trivial in Theorem 2.1. This completes the proof. □

Corollary 2.3 ([1, Theorem 1.6(c)]). *Let H be a commutative $\sigma-$Hopf algebra. Then $(H^{\sigma}, \tilde{\sigma})$ is a braided Hopf algebra with braiding:*

$$\tilde{\sigma}(x, y) = \sum \sigma(y_1, x_1)\sigma^{-1}(x_2, y_2)$$

Proof: Let τ be trivial in Theorem 2.2. This completes the proof. □

Dually, we have the following result:

Theorem 2.4. *Let H be an $\sigma - R-$compatible Hopf algebra, and (H, F) a quasitriangular Hopf algebra. Then $H^{\sigma-R}$ is a quasitriangular Hopf algebra with the following quasitriangular structure:*

$$\tilde{R} = \sum U^{(2)}F^{(1)}R^{(1)} \otimes U^{(1)}F^{(2)}R^{(2)}.$$

3. Classification of $H^{\sigma-R}$ and examples

Let $u \in Hom_k(k, H)$ be convolution invertible with $\epsilon u = \epsilon$, and suppose that H is not only a $R-$Hopf algebra but also a $F-$Hopf algebra satisfying:

$$\sum F^{(1)} \otimes F^{(2)} = \sum (u^{-1} \otimes u^{-1})(R^{(1)} \otimes R^{(2)})\Delta u. \tag{1}$$

This equivalent to

$$\sum G^{(1)} \otimes G^{(2)} = \sum \Delta u^{-1}(U^{(1)} \otimes U^{(2)})(u \otimes u)$$

where G is the inverse of F ($G = \sum G^{(1)} \otimes G^{(2)}$). Now we have

Theorem 3.1. *Let $H^{\sigma-R}$ and $H^{\sigma-F}$ be two Hopf algebras satisfying condition* (1). *Then $H^{\sigma-R} \cong H^{\sigma-F}$ (as coalgebra).*

Proof: Define $\psi : H^{\sigma-R} \longrightarrow H^{\sigma-F}$ by $h \to u^{-1}hu$ and $\varphi : H^{\sigma-F} \longrightarrow H^{\sigma-R}$ by $l \to ulu^{-1}$. We can prove the claim by direct computations. □

Dually, Let $u \in H^*$ with a convolution inverse u^{-1} and $u(1) = 1$, and suppose that H is not only an $\sigma-$Hopf algebra but also a $\tau-$Hopf algebra satisfying:

$$\tau(h, l) = \sum u^{-1}(l_1) \otimes u^{-1}(h_1)\sigma(h_2, l_2)u(h_3, l_3). \tag{2}$$

Then we have the following theorem.

Theorem 3.2. *Let $H^{\sigma - R}$ and $H^{\tau - R}$ be two Hopf algebras satisfying condition* (1). *Then $H^{\sigma - R} \cong H^{\tau - R}$ (as algebra).*

PROOF: Define the maps $\psi : H^{\sigma - R} \longrightarrow H^{\tau - R}$ by $h \to u(h_1)h_2$ and $\varphi : H^{\tau - R} \longrightarrow H^{\sigma - R}$ by $l \to u^{-1}(l_1)l_2$. We can prove the claim by direct computations.

$\square$

Example 1. Let H_4 be the Sweedler's 4-dimensional Hopf algebra ([6]). Define $\sigma : H_4 \otimes H_4 \longrightarrow k$ as follows:

	1	x	y	z
1	1	1	0	0
x	1	-1	0	0
y	0	0	0	0
z	0	0	0	0

and let $R = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + x \otimes 1 - x \otimes x) \in H \otimes H$. It is easy to prove that H_4 is a $\sigma - R$ compatible Hopf algebra.

Example 2. Let $B \bowtie_\tau H$ be the Doi-Takeuchi's Hopf algebra, and $B \otimes_R H$ a Hopf algebra. Let $R = R^{(1)} \otimes R^{(2)} \in B \otimes H$ satisfy $\tau(R^{(1)} c r^{(1)}, h) R^{(2)} \otimes r^{(2)} = \tau(c,h) 1 \otimes 1 = \tau(c, R^{(2)} h r^{(2)}) R^{(1)} \otimes r^{(1)}$. Then $B \otimes H$ is a $[\tau] - Q$ compatible Hopf algebra, where $[\tau](b \otimes g, c \otimes h) = \epsilon(b)\epsilon(h)\tau(c,g)$ for all $b, c \in B, g, h \in H$ and $Q = \sum 1 \otimes R^{(2)} \otimes R^{(1)} \otimes 1$.

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DEPARTMENT OF MATHEMATICS, HENAN NORMAL UNIVERSITY, XINXIANG, HENAN 453002, P.R. CHINA

DEPARTMENT OF MATHEMATICS, QUFU NORMAL UNIVERSITY, QUFU, SHANDONG 273165, P.R. CHINA

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