A note on intermediate differentiability of Lipschitz functions

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Abstract. Let f be a Lipschitz function on a superreflexive Banach space X. We prove that then the set of points of X at which f has no intermediate derivative is not only a first category set (which was proved by M. Fabian and D. Preiss for much more general spaces X), but it is even σ-porous in a rather strong sense. In fact, we prove the result even for a stronger notion of uniform intermediate derivative which was defined by J.R. Giles and S. Sciffer.

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1. Introduction

In this note we show that a theorem of [2] implies a new result on intermediate differentiability of Lipschitz functions.

Let X be a real Banach space. The open ball with center c and radius r is denoted by $B(c, r)$. If f is a Lipschitz function, then the Lipschitz constant of f is denoted by $\mathrm{Lip}(f)$.

If f is a real function on X and $x, v \in X$, then we consider the upper and lower (one-sided) directional derivatives

$$\overline{f}(x, v) = \limsup_{t\to 0+} \frac{f(x+tv) - f(x)}{t} \quad \text{and} \quad \underline{f}(x, v) = \liminf_{t\to 0+} \frac{f(x+tv) - f(x)}{t} .$$

Following [3] we say that $x^* \in X^*$ is an intermediate derivative of a function $f : X \to \mathbb{R}$ at a point $x \in X$ if

$$\underline{f}(x, v) \le (v, x^*) \le \overline{f}(x, v) \quad \text{for every} \quad v \in X.$$

Of course, if f has at x the Gâteaux derivative, then it has also the (unique) intermediate derivative. Therefore Aronszajn's differentiability theorem ([1]) implies that every (locally) Lipschitz function on a separable Banach space has an intermediate derivative at all points except a set E which is null in Aronszajn's sense.

M. Fabian and D. Preiss [3] proved the following theorem.

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Theorem FP. *Suppose that a Banach space Y contains a dense continuous linear image of an Asplund space and that X is a subspace of Y. Then every locally Lipschitz function defined on an open subset Ω of X is intermediate differentiable at every point of $\Omega \setminus A$, where A is a first category set.*

J.R. Giles and S. Sciffer [4] considered the following stronger notion of uniform intermediate differentiability.

Definition 1. A real function f defined on an open subset Ω of a Banach space X is said to be uniformly intermediate differentiable at $x \in \Omega$ if there exists (a "uniform intermediate derivative") $x^* \in X^*$ and a sequence $t_n \searrow 0$ such that

$$\lim_{n\to\infty} \frac{f(x+t_n v) - f(x)}{t_n} = (v, x^*)$$

for each direction $v \in X$ with $\|v\| = 1$.

The following result is proved in [4] using the Preiss deep differentiability theorem of [5].

Theorem GS. *Let X be an Asplund space. Then every locally Lipschitz function defined on an open subset Ω of X is uniformly intermediate differentiable at every point of $\Omega \setminus A$, where A is a first category set.*

To formulate the result of the present note, we need the following definition (cf. [8], p. 327).

Definition 2. Let P be a metric space and $M \subset P$. We say that

(i) M is globally very porous if there exists $c > 0$ such that for every open ball $B(a, r)$ there exists an open ball $B(b, cr) \subset B(a, r) \setminus M$ and

(ii) M is σ-globally very porous if it is a countable union of globally very porous sets.

Remark 1. Every globally very porous set is clearly nowhere dense and thus every σ-globally very porous set is of the first category. It is not difficult to prove that in each Banach space there exists a first category set which is not σ-globally very porous. (For the more difficult result concerning the weaker notion of a σ-porous set see [10].)

Now we can formulate our result.

Theorem. *Let X be a superreflexive Banach space. Then every locally Lipschitz function f defined on an open subset Ω of X is uniformly intermediate differentiable at every point of $\Omega \setminus A$, where A is a σ-globally very porous set.*

By Remark 1, our Theorem is, in the case of a superreflexive X, an improvement of Theorem GS.

A result analogous to Theorem for the weaker notion of (non-uniform) intermediate differentiability is proved in [7] in the case of a separable Banach space X.

In this case the set A can be taken to be "σ-directionally porous". Note that the notions of smallness "σ-globally very porous" and "σ-directionally porous" are incomparable in infinite-dimensional spaces.

We will need also the notion of a very porous set which is clearly weaker than this of a globally very porous set.

Definition 3. Let P be a metric space, $M \subset P$ and $x \in P$. We say that

(i) M is very porous at x if there exist numbers $\delta > 0, \eta > 0$ such that, for each $0 < \rho < \delta$, there exists a ball $B(y, \omega) \subset B(x, \rho) \setminus M$ with $\omega \geq \eta\rho$,

(ii) M is very porous if it is very porous at each of its points and

(iii) M is σ-very porous if it is a countable union of very porous sets.

The basic ingredience of the proof of our Theorem is the following result of [2]. In the terminology of [2], it says that the pair of Banach spaces $(X, \mathbb{R})$ has the "uniform approximation by affine property $(UAAP)$" if X is superreflexive. (Moreover, it is proved in [2] that $(X, \mathbb{R})$ has $(UAAP)$ iff X is superreflexive.)

Theorem BJLPS. *Let X be a superreflexive Banach space. Then for each $\varepsilon > 0$ there exists $c = c(\varepsilon) > 0$ such that for every ball $B(x, \rho)$ in X and every Lipschitz function $f : B(x, \rho) \mapsto \mathbb{R}$ there exist a ball $B(y, \tilde{\rho}) \subset B(x, \rho)$ and an affine function $a : X \mapsto \mathbb{R}$ such that $\tilde{\rho} \geq c\rho$ and*

$$|f(z) - a(z)| \leq \varepsilon\tilde{\rho}\,\mathrm{Lip}(f) \quad \textit{for each} \quad z \in B(y, \tilde{\rho}).$$

We will use also the following relatively easy fact (see [11], Lemma E).

Proposition Z. *Let X be a Banach space and $M \subset X$. Then M is σ-globally very porous iff it is σ-very porous.*

2. Proof of Theorem

Let G_n be the union of all balls $B(c, r) \subset \Omega$ such that $r < 1/n$ and there exists an affine function a on X for which $|f(z) - a(z)| \leq r/n$ whenever $z \in B(c, 2r)$. Put $P_n = \Omega \setminus G_n$ and $A = \bigcup_{n=1}^{\infty} P_n$. It is sufficient to prove that

$$\text{each } P_n \text{ is } \sigma-\text{globally porous and} \tag{1}$$

$$f \text{ has a uniform intermediate derivative at each point of } \Omega \setminus A = \bigcap_{n=1}^{\infty} G_n. \tag{2}$$

First we will prove (1). By Proposition Z, it is sufficient to prove that each P_n is very porous at each point $x \in \Omega$. To this end choose n, x and find $\delta > 0, K > 0$ such that $B(x, \delta) \subset \Omega, \delta < 1/n$ and f is Lipschitz with constant K on $B(x, \delta)$. Now find $c = c(\frac{1}{2nK})$ by Theorem BJLPS and consider an arbitrary $0 < \rho < \delta$.

By the choice of c there exists a ball $B(y,\tilde\rho) \subset B(x,\rho)$ and an affine function a on X such that $\tilde\rho \geq c\rho$ and

$$|f(z) - a(z)| \leq \frac{1}{2nK}\tilde\rho K = \frac{\tilde\rho}{2n} \quad \text{for each} \quad z \in B(y,\tilde\rho).$$

Therefore $B(y,\tilde\rho/2) \subset G_n$ and we see that P_n is very porous at x (with $\eta = c/2$).

To prove (2), suppose that $z \in \bigcap_{n=1}^{\infty} G_n$ is given. Then there exist sequences $(B(c_n, r_n))$ of balls and (a_n) of affine functions on X such that $0 < r_n < 1/n, z \in B(c_n, r_n)$ and

$$|f(y) - a_n(y)| < r_n/n \quad \text{for each} \quad y \in B(c_n, 2r_n). \tag{3}$$

Let $a_n(t) = q_n + x_n^*(t)$, where $q_n \in R$ and x_n^* is a linear function on X. If $v \in X, \|v\| = 1$, then (3) implies

$$\left| \frac{f(z+r_n v) - f(z)}{r_n} - (v, x_n^*) \right| = \left| \frac{f(z+r_n v) - f(z)}{r_n} - \frac{a_n(z+r_n v) - a_n(z)}{r_n} \right| < \frac{2}{n}. \tag{4}$$

Since f is locally Lipschitz, there exist $L > 0$ and $n_0 \in N$ such that $|(v, x_n^*)| < L + 2/n$ whenever $n \geq n_0$ and $\|v\| = 1$. Therefore $(x_n^*)_{n=n_0}^{\infty}$ is a norm bounded sequence in X^*. By the Eberlein-Smulyan theorem we can choose a subsequence $(x_{n_k}^*)_{k=1}^{\infty}$ and $x^* \in X^*$ such that

$$x_{n_k}^* \to x^* \quad \text{in the} \quad w^* - \text{topology}. \tag{5}$$

Put $t_k := r_{n_k}$. Then (4) and (5) clearly imply that

$$\lim_{k\to\infty} \frac{f(z+t_k v) - f(z)}{t_k} = (v, x^*)$$

for each $v \in X, \|v\| = 1$, which completes the proof.

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