On a nonlinear elliptic system: resonance and bifurcation cases

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Abstract. In this paper we consider an elliptic system at resonance and bifurcation type with zero Dirichlet condition. We use a Lyapunov-Schmidt approach and we will give applications to Biharmonic Equations.

Keywords: elliptic system at resonance, bifurcation points, Lyapunov-Schmidt method

Classification: Primary 35J55; Secondary 58J55

1. Introduction

In this paper we shall study the existence of solutions and nonzero solutions of the elliptic system

$$
\text{(S)} \qquad \begin{aligned} -\Delta u &= \lambda u + \delta v + g_1(u,v) - r_1(x) \\ -\Delta v &= \theta u + \gamma v + g_2(u,v) - r_2(x) \end{aligned}
$$

in Ω, where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, subject to Dirichlet boundary conditions $u = v = 0$ on $\partial\Omega$, $r_1(x), r_2(x) \in L^2(\Omega)$, $g_1,\ g_2$ are real valued functions and $\lambda,\ \delta,\ \gamma,\ \theta$ are real numbers.

(S) represents a steady state case of reaction-diffusion systems of interest in biology. Reaction-diffusion systems have been intensively studied during recent years, see [30] where many references can be found.

There exists a decoupling technique, which consists of reducing the system (S) to a single nonlinear equation containing an integral and a differential term. This technique was introduced by Rothe [28], Lazer & McKenna [21] and Brown [6] and has been used thereafter by many authors.

For the resonant case many known techniques used to solve the scalar case can be applied to find solutions and positive solutions. See for example Ahmad, Lazer & Paul [1], Ambrosetti & Mancini [2], Anane [3], Bartolo, Benci & Fortunato [4], Berestycki & De Figueiredo [5], Capozzi, Lupo & Solimini [7], Cesari & Kannan [8], Costa & Magalhães [11], De Figueiredo & Gossez [13], Gossez [15], Innacci & Nkashama [17], Innacci & Nkashama [18], Landesman & Lazer [20], Lupo & Solimini [22], Omari & Zanolin [26], Rabinowitz [27], Schechter [29], Solimini [31], Vargas & Zuluaga [32], Vargas & Zuluaga [33], Zuluaga [34], [35] and the references therein.

The decoupling technique has some obvious shortcomings, for example, it is very difficult to apply to systems with three or more equations. Even, in the case of two equations is too restrictive to give conditions to solve the second equation of (S) for v in terms of u.

Letting

$$U=(u,v),\ -\vec{\Delta}U=\begin{pmatrix}-\Delta u\\ -\Delta v\end{pmatrix},\ A=\begin{pmatrix}\lambda & \delta\\ \theta & \gamma\end{pmatrix},\ G(U)=\begin{pmatrix}g_1(u,v)\\ g_2(u,v)\end{pmatrix},\ R=\begin{pmatrix}r_1(x)\\ r_2(x)\end{pmatrix},$$

we can write (S) as

$$\text{P(R)} \qquad -\vec{\Delta}U=A(U)+G(U)-R.$$

This approach gives us a good meaning of resonance for undecoupling systems. In fact, using the eigenvalues of the matrix A, which are the numbers

$$\zeta=\frac{\lambda+\gamma}{2}-\sqrt{\left(\frac{\lambda-\gamma}{2}\right)^2+\delta\theta},\ \eta=\frac{\lambda+\gamma}{2}+\sqrt{\left(\frac{\lambda-\gamma}{2}\right)^2+\delta\theta},$$

we will be able to give a precise description of the kernel of the operator $-\vec{\Delta}-A$. If we assume that G is bounded it is natural to say that P(R) is a resonant system if there exists $U\in L^2(\Omega)\times L^2(\Omega)$, $U\neq 0$ such that $(-\vec{\Delta}-A)U=0$. If $\delta\theta>0$ then (S) is called a cooperative system, and if $\delta\theta<0$ is called a noncooperative system.

It is known, see [11], and easy to see that the kernel of $-\vec{\Delta}-A$ is nonzero if and only if $A-\lambda_j I$ is singular for some eigenvalue λ_j of the operator $-\Delta$. If $x=(x_1,x_2)\neq(0,0)$ is such that $A(x)=\lambda_j x$ then $U=(x_1\phi_j,x_2\phi_j)\in\operatorname{Ker}(-\vec{\Delta}-A)$.

It is clear that the foregoing presentation is a generalization of resonance of the scalar case known in the literature. In this paper we will consider the case in which $A-\lambda_1 I$ is singular. We shall also study a bifurcation case at $(0,A)$ where A is such that $(\lambda-\lambda_1)(\gamma-\lambda_1)-\delta.\theta=0$. It is a natural generalization of the scalar case where we have bifurcation points at eigenvalues of multiplicity odd. In our case, the *bifurcation point* is a manifold $\mathfrak{M}$ defined by the foregoing equation.

Here we use the fixed point approach and we give applications to biharmonic equations. The references more related to the research in the present work are [11], [14], [32], [33] and [34].

2. Preliminaries and notation

In $E=L^2(\Omega)\times L^2(\Omega)$ we use the induced inner product and norm given by

$$\langle U,\Phi\rangle=\langle u,\phi\rangle_{L^2(\Omega)}+\langle v,\psi\rangle_{L^2(\Omega)},\ \|U\|^2=\|u\|^2_{L^2(\Omega)}+\|v\|^2_{L^2(\Omega)},$$

for $U=(u,v)$ and $\Phi=(\phi,\psi)$. In order to simplify the notation later on, given $U=(u,v)\in E$ with $u=\sum u_j\phi_j$, $v=\sum v_j\phi_j$, where $\{\phi_j\}$ are the eigenfunctions associated with the eigenvalues λ_j, we will say that $U_j = (u_j, v_j) \in \mathbb{R}^2$ are the coordinates of U. Also we will denote $\Phi^j = (\phi_j, \phi_j)$ and we write simply $U = \sum U_j \Phi^j$.

Now we introduce the following bracket

$$[U, \Phi] = \Big(\langle u, \phi \rangle_{L^2(\Omega)}, \langle v, \psi \rangle_{L^2(\Omega)} \Big) \in \mathbb{R}^2,\ U = (u, v),\ \Phi = (\phi, \psi) \in E.$$

By using the foregoing bracket we see that $U_j = [U, \Phi^j]$. It is easy to see that

1. $[U, \Phi] = [\Phi, U]$, for any $U, \Phi \in E$;
2. $[U, \Phi + \Lambda] = [U, \Phi] + [U, \Lambda]$, for any $U, \Phi, \Lambda \in E$;
3. $[\lambda U, \Phi] = \lambda[U, \Phi]$, for any $\lambda \in \mathbb{R}$ and $U, \Phi \in E$;
4. $[(\vec{\Delta})^{-1} U, \Phi] = [U, (\vec{\Delta})^{-1}\Phi]$, for any $U, \Phi \in E$;
5. $U = W$ if and only if $U_j = W_j$ for all $j = 1, 2...$;
6. $[BU, \Phi^j] = B[U, \Phi^j]$, or in this form: $(BU)_j = BU_j$, for any matrix B of order 2×2.

Solutions of P(R). We say that $U \in H_0^1(\Omega) \times H_0^1(\Omega)$ is a solution of P(R) if

$$U = (-\vec{\Delta})^{-1}(AU + G(U) - R). \tag{2.1}$$

It is clear that $(-\vec{\Delta})^{-1} : E \to H_0^1(\Omega) \times H_0^1(\Omega)$ is a linear, selfadjoint, continuous and bijective operator. Also, the embedding $H_0^1(\Omega) \times H_0^1(\Omega) \hookrightarrow E$ is compact, thus $(-\vec{\Delta})^{-1} : E \to E$ is compact, selfadjoint and injective as well.

Since we shall assume that g_1, g_2 are continuous and bounded, the operator G is defined on E with range in E, and it is continuous and bounded. Thus the operator defined by the right hand side of (2.1) is compact.

The Lyapunov-Schmidt method. We will denote by X the subspace of E spanned by Φ^1, that is to say, $X = \{w\Phi^1,\ w \in \mathbb{R}^2\}$. We shall also denote $Y = X^\perp$ on E. Then all $U \in E$ can be written as $U = x + y$, $x \in X$ and $y \in Y$. Let us denote P and Q the projection on X and Y, respectively. Applying P and Q to both sides of (2.1) we obtain a decomposition of it in two equations as follows

$$x = P(-\vec{\Delta})^{-1}(Ax + G(x + y) - R) \tag{2.2}$$

and

$$y = Q(-\vec{\Delta})^{-1}(Ay + G(x + y) - R). \tag{2.3}$$

For each $x \in X$ fixed, we solve (2.3) and the operator solution $y(x)$ will be plugged into (2.2). Thus, the solutions of (2.1) will be of the form $x + y(x)$.

3. Main results

Throughout this paper we shall suppose that $g_i \in C^1$ is bounded and $g_i(0) = 0$ for $i = 1, 2$.

Theorem 3.1. *Suppose that*

(H.1) *G is Lipschitzian with constant k where $k + \|A\| < \lambda_2$;*
(H.2) *λ_1 is an eigenvalue of A with multiplicity two;*
(H.3) *$G'(0)$ is regular.*

Then there are an open set $V \subset E$ and $[a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2$ such that, if $R \in V$ the problem P(R) has a solution, and if $[R, \Phi^1] \notin [a_1, b_1] \times [a_2, b_2]$ then P(R) has no solutions.

Proof: By using (H.1) it is standard to see that for $x \in X$ fixed and $R \in E$ the righthand side of (2.3) is a contraction, thus there exists only one solution $y(x, R) \in Y$ of (2.3). Let $T : X \times E \to Y$ be the function such that $T(x, R)$ is the only fixed point of (2.3). By using (H.1), it is a routine matter to show that T is continuous, $T(0, 0) = 0$ and $T(., R) \in C^1(X, Y)$ for all $R \in E$ and $T(., R)$ is bounded in C^1 norm. See [2] or [32] for details.

Now we plug $T(x, R)$ in (2.2) and our problem is reduced to find solutions of

$$x = P(-\vec{\Delta})^{-1}(Ax + G(x + T(x, R)) - R). \tag{3.1}$$

Let $x = w\Phi^1$ with some $w \in \mathbb{R}^2$. The condition (H.2) ensures any $w \in \mathbb{R}^2$ is an eigenvector of A corresponding to λ_1. Hence, by properties 4 and 5 of bracket, noting that $(-\vec{\Delta})^{-1}\Phi^1 = \frac{1}{\lambda_1}\Phi^1$, and by (H.2), (3.1) is equivalent to

$$[R, \Phi^1] = [G(w\Phi^1 + T(w\Phi^1, R)), \Phi^1]. \tag{3.2}$$

Let $\mathfrak{F}(w, R) = [G(w\Phi^1 + T(w\Phi^1, R)) - R, \Phi^1]$. Then $\mathfrak{F}(0, 0) = 0$ and a calculation tells us that $\mathfrak{F}'_w(0, 0) = G'(0)$. By (H.3) and the Implicit Function Theorem there exist an open neighborhood V of $R = 0$ and a continuous function $\psi : V \to \mathbb{R}^2$ such that

$$[R, \Phi^1] = [G(\psi(R)\Phi^1 + T(\psi(R)\Phi^1, R)), \Phi^1]. \tag{3.3}$$

By (3.3) it is clear that if $R \in V$ then (3.1) has a solution $x = w\Phi^1$ and therefore P(R) has a solution in the form $w\Phi^1 + T(w\Phi^1, R)$.

On the other hand, let

$$a_i = \inf_{\substack{w\in\mathbb{R}^2 \\ R\in E}} \int_\Omega g_i(w\Phi^1 + T(w\Phi^1, R))\phi_1,\ i = 1, 2$$

and

$$b_i = \sup_{\substack{w\in\mathbb{R}^2 \\ R\in E}} \int_\Omega g_i(w\Phi^1 + T(w\Phi^1, R))\phi_1,\ i = 1, 2.$$

Since G is bounded, a_i, b_i are finite numbers. Then by (3.3) we conclude our second assertion. $\square$

Remark. Theorem 3.1 is a generalization of the scalar case. In that case the problem has a solution if $\int_\Omega R\phi_1 \in (a_1, b_1)$ and it has not solution if $\int_\Omega R\phi_1 \notin [a_1, b_1]$. See [2] or [32].

If we change the hypothesis (H.2) in Theorem 3.1 we can get a theorem in the same spirit as that in the scalar case.

Let us suppose that A has two eigenvalues, λ_1 and μ. Let us denote $\Lambda = (\sin\alpha, \cos\alpha)$ and $\Pi = (\sin\beta, \cos\beta)$ the eigenvectors associated with λ_1 and μ, respectively. Then we have the following

Theorem 3.2. *Let us assume* (H.1) *and*

(H.4) *A has two eigenvalues, λ_1 and μ such that $(\mu - \lambda_1) \neq 0$.*
(H.5) *The following inequality holds*

$$\frac{k}{|\mu - \lambda_1||p|}\left\{1 + \frac{k(|\sin\beta| + |\cos\beta|)}{\lambda_2 - \|A\| - k}\right\} < 1,$$

where p is $\sin\beta$ or $\cos\beta$.

Then we have: If $p = \sin\beta \neq 0$ there exists $[c_2, d_2]$ finite such that if $\int_\Omega r_2\phi_1 \in (c_2, d_2)$ then the problem P(R) has a solution, and if $\int_\Omega r_2\phi_1 \notin [c_2, d_2]$ then the problem P(R) has no solutions. We have a similar result if $p = \cos\beta \neq 0$.

Proof: Suppose that $p = \sin\beta \neq 0$, since $\{\Lambda,\ \Pi\}$ is linearly independent, $w \in \mathbb{R}^2$ can be written as $w = \boldsymbol{x}\Lambda + \boldsymbol{y}\Pi$, for some $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}$ and we can write (3.1) with $x = (\boldsymbol{x}\Lambda + \boldsymbol{y}\Pi)\Phi^1$ as

$$(\boldsymbol{x}\Lambda + \boldsymbol{y}\Pi)\Phi^1 = P(-\vec{\Delta})^{-1}(A(\boldsymbol{x}\Lambda + \boldsymbol{y}\Pi)\Phi^1 + G(w\Phi^1 + T(w\Phi^1, R)) - R),$$

which can be written as

$$[R, \Phi^1] = [\boldsymbol{y}(\mu - \lambda_1)\Pi\Phi^1 + G(w\Phi^1 + T(w\Phi^1, R), R), \Phi^1].$$

That is to say

$$\int_\Omega r_1\phi_1 = (\mu - \lambda_1)\boldsymbol{y}\sin\beta + \int_\Omega g_1(w\Phi^1 + T(w\Phi^1, R))\phi_1 \tag{3.4.a}$$

and

$$\int_\Omega r_2\phi_1 = (\mu - \lambda_1)\boldsymbol{y}\cos\beta + \int_\Omega g_2(w\Phi^1 + T(w\Phi^1, R))\phi_1. \tag{3.4.b}$$

Now, by (H.5), for each $\boldsymbol{x} \in \mathbb{R}$ fixed

$$f(\boldsymbol{y}) = \frac{1}{(\mu - \lambda_1)\sin\beta}\left\{\int_\Omega (r_1 - g_1((\boldsymbol{x}\Lambda + \boldsymbol{y}\Pi)\Phi^1 + T((\boldsymbol{x}\Lambda + \boldsymbol{y}\Pi)\Phi^1, R)))\phi_1\right\}$$

is a contraction. Then, for each $\boldsymbol{x} \in \mathbb{R}$ fixed, (3.4.a) has only one solution $\boldsymbol{y} = t(\boldsymbol{x}, R)$. Now we can argue as in [2] or [32] and see that $t : \mathbb{R} \times E \to \mathbb{R}$ is continuous, $t(0,0) = 0$, $t(., R) \in C^1$ and $t(., R)$ is bounded in C^1.

Finally we plug $t(\boldsymbol{x}, R)$ in (3.4.b) and we obtain our assertion, where

$$c_2 = \inf_{\boldsymbol{x}\in\mathbb{R}} \left\{ (\mu - \lambda_1) \cos\beta . t(\boldsymbol{x}, R) + \int_\Omega g_2(w\Phi^1 + T(w\Phi^1, R))\phi_1 \right\}$$

and

$$d_2 = \sup_{\boldsymbol{x}\in\mathbb{R}} \left\{ (\mu - \lambda_1) \cos\beta . t(\boldsymbol{x}, R) + \int_\Omega g_2(w\Phi^1 + T(w\Phi^1, R))\phi_1 \right\}.$$

□

Nonzero solutions. Now we consider the homogeneous case of system (S). That is: $r_1(x) = r_2(x) = 0$. Then we have the following

Theorem 3.3. *Let us assume* (H.1) *and suppose that*

(H.6) *A has two eigenvalues, λ_1 and μ such that $(\mu - \lambda_1) > 2$;*
(H.7) $D_1 g_1(0)\{D_2 g_2(0) + \mu - \lambda_1\} - D_1 g_2(0) D_2 g_1(0) > 0$.

Then system (S), where $r_1(x) = r_2(x) = 0$, has at least a nonzero solution.

Proof: As in Theorem 3.1, our problem can be reduced to find solutions of

$$x = P(-\vec{\Delta})^{-1}(Ax + G(x + T(x))). \tag{3.5}$$

Now, (3.5) is equivalent to

$$0 = P(-\vec{\Delta})^{-1}([A - \lambda_1]x + G(x + T(x))). \tag{3.6}$$

Let $x = w\Phi^1$ with some $w \in \mathbb{R}^2$. Hence, the properties 4 and 5 of the bracket and (H.6) tell us that (3.6) is equivalent to

$$0 = [(A - \lambda_1)w\Phi^1 + G(w\Phi^1 + T(w\Phi^1)), \Phi^1]$$

that we can put as

$$0 = \mathbf{A}w + \mathbf{G}w, \tag{3.7}$$

where $\mathbf{G}w = [G(w\Phi^1 + T(w\Phi^1)), \Phi^1]$ and $\mathbf{A}w = [(A - \lambda_1)w\Phi^1, \Phi^1]$. It is important to note that the eigenvalues of $\mathbf{A}$ are 0 and $\mu - \lambda_1$.

The condition (H.7) tells us that

$$\operatorname{ind}[-(\mathbf{A} + \mathbf{G}), 0] = 1. \tag{3.8}$$

Now, since $\mathbf{G}$ is bounded, $\mathbf{A}+\mathbf{G}-I$ is asymptotically linear to $\mathbf{A}-I$ and its eigenvalues are -1 and $\mu-\lambda_1-1$. Then by Theorem 21.2 of [19] we have

$$d_B[I-\{\mathbf{A}+\mathbf{G}-I\},B(0,\rho),0]=(-1)^1=-1, \tag{3.9}$$

for ρ large enough. Here, d_B denotes the Brouwer degree.

Now, by using (H.6), we can see that $2I-\{\mathbf{A}+\mathbf{G}\}$ and $-\{\mathbf{A}+\mathbf{G}\}$ are homotopically equivalent on $\partial B(0,\rho)$ for ρ large enough. To see that it is sufficient to consider the homotopy

$$H(t,u)=2f(t)u-g(t)(\mathbf{A}+\mathbf{G}), 0\le t\le 1,$$

where, it is clear, the case $t=0$ can be omitted and the functions f, g can be picked such that $f(1)=g(1)=1$, $f(0)=0$, $g(0)=1$ and $f<g$ for $t\neq 1$. If we suppose that $H(t,u)=0$ on $\partial B(0,\rho)$ for ρ large enough we get that $\mathbf{A}$ has a positive eigenvalue less than 2. This contradiction shows our assertion. Then by (3.9) we get

$$d_B[-\{\mathbf{A}+\mathbf{G}\},B(0,\rho),0]=(-1)^1=-1. \tag{3.10}$$

By (3.8), (3.10) and the domain decomposition property of the Brouwer degree theory there exists $\alpha\in\mathbb{R}^2$, $\alpha\neq 0$ such that $\alpha\Phi_1+T(\alpha\Phi_1)$ is a nonzero solution of system (S) in the case $r_1(x)=r_2(x)=0$. □

Example. As an application of Theorem 3.3 we have the following example: The system

$$\begin{aligned}-u''&=4u+\sin v\\ -v''&=\alpha v-\sin u,\end{aligned}$$

with $u(0)=v(0)=u(\frac{\pi}{2})=v(\frac{\pi}{2})=0$ and $6<\alpha<12$ has at least a nonzero solution.

4. Bifurcation results

The (H.2) condition tells us that $A\in\mathfrak{M}$ with

$$\mathfrak{M}=\{(\lambda,\gamma,\delta,\theta)\in\mathbb{R}^4;(\lambda-\lambda_1)(\gamma-\lambda_1)-\delta\theta=0\}.$$

It is easy to see that $\mathfrak{M}-(\lambda_1,\lambda_1,0,0)$ is a C^1 manifold. Our main theorem is

Theorem 4.1. *Suppose that* (H.1) *and* (H.8) *hold with*

(H.8) $G'(0)=0$.

Then for any matrix $A\in M_{2\times 2}$ for which one and only one of the conditions (H.4) *or* (H.2) *holds,* $(0,A)$ *is a bifurcation point of*

$$-\vec{\Delta}U=A(U)+G(U). \tag{P(0)}$$

Proof: First we shall suppose that (H.4) holds. Let $B = A + t\lambda_1 I$, $t \in \mathbb{R}$. For $|t|$ small enough, the (H.1) condition holds for our matrix B. Then the solutions of $-\vec{\Delta} U = B(U) + G(U)$ can be reduced to find solutions of

$$x = P(-\vec{\Delta})^{-1}(Bx + G(x + T(x, B))), \tag{4.1}$$

as we did in the proof of Theorem 3.1. It is important to remark that $T : X \times M_{2\times 2} \to Y$ is C^1 and, by (H.1), $T(0, .) = 0$.

Let $x = w\Phi^1$ with some $w \in \mathbb{R}^2$. Hence, by properties 4 and 5 of the bracket and by (H.4), the equation (4.1) is equivalent to

$$0 = [t\lambda_1 w\Phi^1 + (\mu - \lambda_1)\boldsymbol{P}(w\Phi^1) + G(w\Phi^1 + T(w\Phi^1, B)), \Phi^1], \tag{4.2}$$

where $\boldsymbol{P}$ is the projection onto the eigenspace associated with μ. By (H.8) we conclude that there exists $\widetilde{G}(w, t)$ such that $G(x + T(x, A + t\lambda_1 I)) = \widetilde{G}(x, t).x$ with $\widetilde{G}(0, t) = 0$. Now, since $Y = X^\perp$ and $T(w\Phi^1, C) \in Y$, for any $w \in \mathbb{R}^2$ and any matrix C of order 2×2, we see that $[T(w\Phi^1, C), z\Phi^1] = 0$, for all $z, w \in \mathbb{R}^2$ and for any matrix C of order 2×2. Hence we have

$$\begin{aligned}
&\frac{d}{dt}[G(w\Phi^1 + T(w\Phi^1, A + t\lambda_1 I)), \Phi_1]\Big|_{t=0} \\
&= [G'(w\Phi^1 + T(w\Phi^1, A)) \circ T_2'(w\Phi^1, A)(w\Phi^1, \lambda_1 I), \Phi^1] \\
&= 0.
\end{aligned}$$

By the foregoing equation we obtain

$$\widetilde{G}_t(w\Phi^1, 0) = 0. \tag{4.3}$$

Now, (4.2) is equivalent to

$$0 = t\lambda_1 w + (\mu - \lambda_1)\boldsymbol{P}(w) + \widetilde{G}(w\Phi^1, t)w. \tag{4.4}$$

Now, let

$$\boldsymbol{H} = \left\{ w \in \mathbb{R}^2; \boldsymbol{P}(w) = 0 \right\}.$$

It is clear that our problem can be reduced to find zeros of

$$0 = t\lambda_1 w + \widetilde{G}(w\Phi^1, t)w, \quad w \in \boldsymbol{H}.$$

These zeros can be found if we find zeros of

$$0 = t\lambda_1 + \widetilde{G}(w\Phi^1, t), \quad w \in \boldsymbol{H}.$$

If we denote $F(w, t) = t\lambda_1 + \widetilde{G}(w\Phi^1, t)$ we see that $F(\Theta, 0) = 0$ and, by (4.3), $F_t(\Theta, 0) = \lambda_1$, where $\Theta = (0, 0) \in \mathbb{R}^2$. The Implicit Function Theorem tells us that there are a neighborhood V of Θ ($V \subset \boldsymbol{H}$) and a continuous function $\Psi : V \to \mathbb{R}$ such that $\Psi(\Theta) = 0$ and $F(w, \Psi(w)) = 0$. Then, for all $w \in V$,

$$0 = \Psi(w)\lambda_1 + \widetilde{G}(w, \Psi(w)).$$

Now, since Ψ is unique, $\Psi \neq 0$, there exists a sequence $\{w_n\} \subset V$ such that $w_n \to 0$, $w_n \neq 0$ and $\Psi(w_n) \to 0$, $\Psi(w_n) \neq 0$. It is clear that

$$\{s_n = \Psi(w_n)\lambda_1 w_n \Phi^1 + T(w_n\Phi^1, A + \Psi(w_n)\lambda_1 I)\}$$

is a sequence of nonzero solutions of $P(0)$ such that $s_n \to 0$.

Finally, if we assume that the (H.2) holds then $\mu - \lambda_1 = 0$ and (4.4) will be $0 = t\lambda_1 w + \widetilde{G}(w\Phi^1, t)w$ with $w \in \mathbb{R}^2$. Now, we argue as we did before, in this case without the restriction $w \in \boldsymbol{H}$, and the Theorem is proved. □

Remark. As a consequence of Theorem 4.1 we can consider an interesting example common in the literature. Suppose that $r_1 = r_2 = 0$ and $\lambda_1 < \lambda < \lambda_2$, $\delta = 0$ and $\gamma = \lambda_1$. Also suppose that $g_1(u, v) = 0$ and $g_2(u, v) = g_2(v)$. It is well known that $u = 0$ and (S) can be reduced to the scalar problem

$$-\Delta v = \lambda_1 v + g_2(v). \tag{4.5}$$

Then we have the following

Corollary 4.2. *Suppose* (H.1), (H.4) *and* (H.8) (*in this case* (H.8) *has the form* $g_2'(0) = 0$). *Then* $(0, \lambda_1)$ *is a bifurcation point of* (4.5).

This result is a particular case of Theorem 5.1 in [10, p. 188].

Remark. In the scalar case (4.5) the condition (H.4) can be weakened. See for example [32, p. 251], where a bifurcation point of supercritical type was considered.

5. Biharmonic equations

Theorem 3.2 can be applied to biharmonic equations under Navier and Dirichlet conditions. For example

$$-\Delta^2 u = -\lambda_1^2 u + g_2(u) - r_2(x), \tag{5.1}$$

where $u = \Delta u = 0$ on $\partial\Omega$.

Indeed, (5.1) can be stated as

$$\begin{aligned} -\Delta u &= 0u + (-1)v \\ -\Delta v &= -\lambda_1^2 u + 0v + g_2(u) - r_2(x). \end{aligned}$$

Here the matrix A is $A = \begin{pmatrix} 0 & -1 \\ -\lambda_1^2 & 0 \end{pmatrix}$ and its eigenvalues are: λ_1 and $-\lambda_1$. Then, as Theorem 3.2, we have the following

Theorem 5.1. *Suppose* (H.1) *and* (H.5). *Then there exists* $[c_2, d_2]$ *finite such that if* $\int_\Omega r_2\phi_1 \in (c_2, d_2)$ *the problem* (5.1) *has a solution, and if* $\int_\Omega r_2\phi_1 \notin [c_2, d_2]$ *the problem* (5.1) *has no solutions.*

PROOF: Note that all conditions of Theorem 3.2 hold. □

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References

[1] Ahmad S., Lazer A., Paul J., *Elementary critical point theory and perturbation of elliptic boundary value problems at resonance*, Indiana Univ. Math. J. **25** (1976), 933–944.

[2] Ambrosetti A., Mancini G., *Existence and multiplicity results for nonlinear elliptic problems with linear part at resonance. The case of the simple eigenvalue*, J. Diff. Equations **28** (1978), 220–245.

[3] Anane A., *Etude des valeurs propers et la resonance pour le operateur P-Laplacian*, Ph.D. Thesis, Univ. Bruxelles, 1988.

[4] Bartolo P., Benci V., Fortunato D., *Abstract critical point Theorems and applications to some nonlinear problems with strong resonance*, Nonlinear Analysis T.M.A. **7** (1983), no. 9, 981–1012.

[5] Berestycki H., Defigueiredo D., *Double resonance in semilinear elliptic problems*, Comm. Partial Diff. Equations **6** (1981), 91–120.

[6] Brown K.J., *Spatially inhomogeneous steady-state solutions for systems of equations describing interacting populations*, J. Math. Anal. Appl. **95** (1983), 251-264.

[7] Capozzi A., Lupo D., Solimini S., *On the existence of a nontrivial solution to nonlinear problem at resonance*, Nonlinear Analysis T.M.A. **13** (1989), no. 2, 151–163.

[8] Cesari L., Kannan R., *Qualitative study of a class of nonlinear boundary value problems at resonance*, J. Diff. Equations **56** (1985), 63–81.

[9] Chiappinelli R., Mawhin J., Nugari R., *Bifurcation from infinity and multiple solutions for some Dirichlet problems with unbounded nonlinearities*, Nonlinear Analysis T.M.A, in press.

[10] Chow S., Hale J., *Methods of Bifurcation Theory*, Springer Verlag, 1982.

[11] Costa D., Magalhães, *A variational approach to subquadratic perturbations of elliptic systems*, J. Diff. Equations **111** (1994), no. 1, 103–122.

[12] De Figueiredo D., Chiappinelli R., *Bifurcation from infinity and multiple solutions for an elliptic system*, Differential and Integral Equations **6** (1993), no. 4, 757–771.

[13] De Figueiredo D., Gossez J., *Resonance below the first eigenvalue for a semilinear elliptic problem*, Math. Ann. **281** (1988), 589–610.

[14] De Figueiredo D., Mitidieri E., *A maximum principle for an elliptic system and applications to semilinear problems*, Siam. J. Math. Anal. **17** (1986), 836–849.

[15] Gossez J., *Some nonlinear differential equations at resonance at first eigenvalue*, Conf. Sem. Mat. Univ Bari **167** (1979), 355–389.

[16] Hernández J., *Maximum Principles and Decoupling for Positive Solutions of Reaction-Diffusion Systems*, Oxford University Press, K.J. Brown and A. Lacey eds, 1990, pp. 199–224.

[17] Iannacci R., Nkashama M., *Nonlinear boundary value problems at resonance*, Nonlinear Analysis T.M.A. **11** (1987), 455–473.

[18] Iannacci R., Nkashama M., *Nonlinear second order elliptic partial differential equations at resonance*, Report 87–12, Memphis State University, 1987.

[19] Krasnosels'kii M., Zabreico F., *Geometrical Methods of Nonlinear Analysis*, Springer Verlag, 1984.

[20] Landesman E., Lazer A., *Nonlinear perturbation of elliptic boundary value problems at resonance*, J. Math. Mech. **19** (1970), 609–623.

[21] Lazer A., McKenna P.J., *On steady-state solutions of a system of reaction-diffusion equations from biology*, Nonlinear Analysis T.M.A. **6** (1982), 523–530.

[22] Lupo D., Solimini S., *A note on a resonance problem*, Proc. Royal Soc. Edinburgh **102 A** (1986), 1–7.

[23] Mawhin J., *Bifurcation from infinity and nonlinear boundary value problems, in ordinary and partial differential equations*, vol. II, Sleeman and Jarvis eds, Longman, Ifarlow, 1989, pp. 119–129.

[24] Mawhin J., Schmitt K., *Landesman-Lazer type problems at an eigenvalue of odd multiplicity*, Results in Math. **14** (1988), 138–146.

[25] Mawhin J., Schmitt K., *Nonlinear eigenvalue problems with the parameter near resonance*, Ann. Polon. Math. **51** (1990), 241–248.

[26] Omari P., Zanolin F., *A note on nonlinear oscillations at resonance*, Acta Math. Sinica **3** (1987), 351–361.

[27] Rabinowitz P., *Minimax methods in critical point theory with applications to differential equations*, CBMS 65 Regional Conference Series in Math, A.M.S., 1986.

[28] Rothe F., *Global existence of branches of stationary solutions for a system of reaction-diffusion equations from biology*, Nonlinear Analysis T.M.A. **5** (1981), 487–498.

[29] Schechter M., *Nonlinear elliptic boundary value problems at resonance*, Nonlinear Analysis T.M.A. **14** (1990), no. 10, 889–903.

[30] Smoller J., *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, 1983.

[31] Solimini S., *On the solvability of some elliptic partial differential equations with the linear part at resonance*, J. Math. Anal. Appl. **117** (1986), 138–152.

[32] Vargas C., Zuluaga M., *On a nonlinear Dirichlet problem type at resonance and bifurcation*, Partial Differential Equations (1992), Pitmann Research Notes in Math. Series 273, pp. 248–252.

[33] Vargas C., Zuluaga M., *A nonlinear elliptic problem at resonance with a nonsimple eigenvalue*, Nonlinear Analysis T.M.A. (1996), 711–721.

[34] Zuluaga M., *A nonlinear elliptic system at resonance*, Dynamic Systems and Applications **3** (1994), no. 4, 501–510.

[35] Zuluaga M., *Nonzero solutions of a nonlinear elliptic system at resonance*, Nonlinear Analysis T.M.A **31** (1996), no. 3/4, 445–454.

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