Fréchet directional differentiability and Fréchet differentiability

J.R. Giles, Scott Sciffer

Abstract. Zajíček has recently shown that for a lower semi-continuous real-valued function on an Asplund space, the set of points where the function is Fréchet subdifferentiable but not Fréchet differentiable is first category. We introduce another variant of Fréchet differentiability, called Fréchet directional differentiability, and show that for any real-valued function on a normed linear space, the set of points where the function is Fréchet directionally differentiable but not Fréchet differentiable is first category.

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A real-valued function F on an open subset A of a normed linear space X is said to be *Fréchet differentiable* at $x \in A$ if there exists a continuous linear functional $F'(x)$ on X where, given $\epsilon > 0$ there exists a $\delta(\epsilon, x) > 0$ such that

$$|F(x+y) - F(x) - F'(x)(y)| < \epsilon\|y\| \quad \text{for all} \quad y \in X, \ \|y\| < \delta.$$

In determining Fréchet differentiability properties, interest has focused on variants of Fréchet differentiability. The function F is said to be *Fréchet subdifferentiable* at $x \in A$ if there exists a continuous linear functional f on X where, given $\epsilon > 0$ there exists a $\delta(\epsilon, x) > 0$ such that

$$F(x+y) - F(x) - f(y) > -\epsilon\|y\| \quad \text{for all} \quad y \in X, \ \|y\| < \delta.$$

In particular, Borwein and Preiss proved that when X is a Banach space with an equivalent norm Fréchet differentiable away from the origin, a lower semi-continuous function F on an open subset A of X is densely Fréchet subdifferentiable, [BP, p. 521]. Recently Zajíček proved that when X is an Asplund space, a lower semi-continuous function F on an open subset A of X has the property that the set of points where F is Fréchet subdifferentiable but not Fréchet differentiable is first category in A, [Z2, p. 485].

In this paper we study another variant of Fréchet differentiability. Given a real-valued function F on an open subset A of a normed linear space X, we say that F has a *right-hand derivative* at $x \in A$ in the direction $v \in X$ if

$$F'_+(x)(v) = \lim_{\lambda \to 0^+} \frac{F(x+\lambda v) - F(x)}{\lambda}$$

exists. Clearly $F'_+(x)(v)$ is positively homogeneous in v. We say F is *directionally differentiable* at $x \in A$ if $F'_+(x)(v)$ exists in every direction $v \in X$ and is a continuous function in v. If $F'_+(x)(v)$ is also linear in v then we say that F is *Gâteaux differentiable* at x. We note that although F may have a right-hand derivative at $x \in A$ in every direction $v \in X$, $F'_+(x)(v)$ need not be continuous in v, even if F is continuous at x.

Example. Consider F on $\mathbf{R}^2$ defined in polar coordinates by

$$F(r,\theta) = \begin{cases} \cos\theta \sin(r/\cos\theta) & \text{for } \cos\theta \neq 0 \text{ and } r \neq 0 \\ 0 & \text{for } \cos\theta = 0 \text{ or } r = 0. \end{cases}$$

Now at the origin

$$\frac{\partial F}{\partial r} = \begin{cases} 1 & \text{when } \cos\theta \neq 0 \\ 0 & \text{when } \cos\theta = 0. \end{cases}$$

□

However, if a locally Lipschitz function has a right-hand derivative at a point in every direction, then it is directionally differentiable at the point. This is the implication of the following well known result whose proof is included for the sake of completeness.

Proposition 1. *Consider a locally Lipschitz function ψ on an open subset A of a normed linear space X. Given $x \in A$, if $\psi'_+(x)(v)$ exists for all $v \in X$ then $\psi'_+(x)(v)$ is Lipschitz in v.*

Proof: Since ψ is locally Lipschitz on A, there exists a $K > 0$ and a $\delta > 0$ such that

$$|\psi(y) - \psi(z)| \leq K\|y - z\| \quad \text{for all } \; y, z \in B(x;\delta) \cap A.$$

Given $u \in X$, $\|u\| \leq 1$ and $\epsilon > 0$ there exists a $0 < \delta_1 < \delta$ such that

$$\begin{aligned} \psi'_+(x)(u) - \epsilon &< \frac{\psi(x+\lambda u) - \psi(x)}{\lambda} \quad \text{for } \; 0 < \lambda < \delta_1 \\ &\leq \frac{\psi(x+\lambda v) - \psi(x)}{\lambda} + K\|u - v\| \\ &\qquad \text{for } \; v \in X, \; \|v\| \leq 1 \text{ and } \; 0 < \lambda < \delta_1. \end{aligned}$$

But there exists $0 < \delta_2 < \delta_1$ such that

$$\frac{\psi(x+\lambda v) - \psi(x)}{\lambda} < \psi'_+(x)(v) + \epsilon \quad \text{for } \; 0 < \lambda < \delta_2.$$

And so

$$\psi'_+(x)(u) - \epsilon < \psi'_+(x)(v) + \epsilon + K\|u - v\| \quad \text{for all } \; u, v \in X, \; \|u\|, \|v\| \leq 1.$$

This holds for all $\epsilon > 0$ and so we conclude that

$$\psi'_+(x)(u) \le \psi'_+(x)(v) + K\|u - v\|$$

and so

$$|\psi'_+(x)(u) - \psi'_+(x)(v)| \le K\|u - v\| \quad \text{for all} \quad u, v \in X,$$

since $\psi'_+(x)(v)$ is positively homogeneous in v. □

For a real-valued function F on an open subset A of a normed linear space X we say that F is *Fréchet directionally differentiable* at $x \in A$ if F is directionally differentiable at x and given $\epsilon > 0$ there exists $\delta(\epsilon, x) > 0$ such that

$$\left| \frac{F(x + \lambda v) - F(x)}{\lambda} - F'_+(x)(v) \right| < \epsilon \quad \text{for all} \quad 0 < \lambda < \delta \quad \text{and all} \quad v \in X, \ \|v\| \le 1.$$

Of course F is Fréchet differentiable at $x \in A$ if it is Fréchet directionally differentiable at x and $F'_+(x)(v)$ is linear in v.

It is of interest to see how Fréchet directional differentiability and Fréchet subdifferentiability relate. A continuous convex function ϕ on an open convex subset A of a normed linear space X has a subgradient at each point of its domain; that is, given $x \in A$ there exists a continuous linear functional f on X such that

$$\phi(x + y) - \phi(x) \ge f(y) \quad \text{for all} \quad y \in X, \quad \text{[Ph, p. 7]}.$$

So ϕ is Fréchet subdifferentiable at each point in its domain. But also $\phi'_+(x)(y)$ exists and is a continuous sublinear functional in y at each point $x \in A$, [Ph, p. 2], so ϕ is directionally differentiable at each point of its domain. However, the norm is always Fréchet directionally differentiable at the origin, but it need not be Fréchet directionally differentiable at any other point; there exists on ℓ_1 an equivalent norm which is Gâteaux differentiable away from the origin but which is nowhere Fréchet differentiable, [Ph, p. 86].

More generally, for a real-valued function F on an open subset A of a normed linear space X, if F is Fréchet directionally differentiable at $x \in A$ then given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$F(x + y) - F(x) > F'_+(x)(y) - \epsilon\|y\| \quad \text{for all} \quad y \in X, \ \|y\| < \delta,$$

and if $F'_+(x)(y)$ is also sublinear in y then any subgradient f of $F'_+(x)$ at x satisfies

$$F(x + y) - F(x) > f(y) - \epsilon\|y\| \quad \text{for all} \quad y \in X, \ \|y\| < \delta,$$

that is, F is Fréchet subdifferentiable at x. However, if F is Fréchet directionally differentiable at x then it is not necessarily Fréchet subdifferentiable at x; for the norm $\|.\|$ on X, $F = -\|.\|$ is Fréchet directionally differentiable at 0 but is not Fréchet subdifferentiable at 0.

We now establish for Fréchet directional differentiability a general result comparable to that of Zajíček for Fréchet subdifferentiability, [Z2, p. 485].

Theorem 2. *Given a real-valued function F on an open subset A of a normed linear space X, the set of points $W \subset A$ where F is Fréchet directionally differentiable, but not Fréchet differentiable, is first category in A.*

Proof: For each $n, p \in \mathbf{N}$ consider the set $W_{n,p}$ consisting of those points in A where F is Fréchet directionally differentiable and

(i)
$$\left|\frac{F(x+\lambda v)-F(x)}{\lambda} - F'_+(x)(v)\right| < \frac{1}{p}$$
$$\text{for all } \ 0 < \lambda \leq \frac{2}{n} \ \text{ and all } \ v \in X, \ \|v\| \leq 1,$$

(ii)
$$\text{there exist } u, v \in X, \|u\|, \|v\| \leq 1 \ \text{ and}$$
$$|F'_+(x)(u) + F'_+(x)(v) - F'_+(x)(u+v)| > \frac{16}{p}\,.$$

Now $W = \cup_{n,p\geq 2} W_{n,p}$ and we show that for each $n, p \geq 2$, $W_{n,p}$ is nowhere dense in A.

Suppose on the contrary that for some $n, p \geq 2$, $W_{n,p}$ is dense in some open subset U of A. We may assume that $0 \in W_{n,p} \cap U$ and $F(0) = 0$. We examine estimates of F near 0. For $m > n$, by (i), we have

$$|F(\frac{1}{n}u) - \frac{1}{n}F'_+(0)(u)| < \frac{1}{np}, \quad |F(\frac{1}{m}v) - \frac{1}{m}F'_+(0)(v)| < \frac{1}{mp} \ \text{ and}$$
$$|F(\frac{1}{m}(u+v)) - \frac{1}{m}F'_+(0)(u+v)| < \frac{1}{mp}\,.$$

We now choose m sufficiently large that $B(0;2/m) \subset U$ and, to satisfy the continuity of $F'_+(0)$, such that

$$|F'_+(0)(\frac{1}{n}u) - F'_+(0)(x + \frac{1}{n}u)| < \frac{1}{np} \ \text{ for all } \ x \in B(0;2/m).$$

Since $W_{n,p}$ is dense in U, by the continuity of $F'_+(0)$, there exists an $x_m \in B(0;2/m) \cap W_{n,p}$ such that both

$$|F'_+(0)(\frac{1}{m}v) - F'_+(0)(x_m)| < \frac{1}{mp} \ \text{ and}$$
$$|F'_+(0)(\frac{1}{m}(u+v)) - F'_+(0)(x_m + \frac{1}{m}u)| < \frac{1}{mp}\,.$$

However, by property (i), $|F(x_m) - F'_+(0)(x_m)| < (1/p)\|x_m\|$ and so

$$|F(x_m) - F'_+(0)(\frac{1}{m}v)| \leq |F(x_m) - F'_+(0)(x_m)| + |F'_+(0)(x_m) - F'_+(0)(\frac{1}{m}v)|$$
$$< \frac{1}{p}\|x_m\| + \frac{1}{mp} < \frac{3}{mp}\,. \qquad \ldots(a)$$

Similarly, when $m \geq 2n$ we have

$$\begin{aligned}
&|F(x_m + \frac{1}{m}u) - F'_+(0)(\frac{1}{m}(u+v))| \\
&\leq |F(x_m + \frac{1}{m}u) - F'_+(0)(x_m + \frac{1}{m}u)| + |F'_+(0)(x_m + \frac{1}{m}u) - F'_+(0)(\frac{1}{m}(u+v))| \\
&< \frac{1}{p}\|x_m + \frac{1}{m}u\| + \frac{1}{mp} < \frac{4}{mp} \quad \text{since} \quad \|x_m + \frac{1}{m}u\| \leq \frac{3}{m} \leq \frac{2}{n}\,. \qquad \ldots \text{(b)}
\end{aligned}$$

Also

$$\begin{aligned}
&|F(x_m + \frac{1}{n}u) - F'_+(0)(\frac{1}{n}u)| \\
&\leq |F(x_m + \frac{1}{n}u) - F'_+(0)(x_m + \frac{1}{n}u)| + |F'_+(0)(x_m + \frac{1}{n}u) - F'_+(0)(\frac{1}{n}u)| \\
&< \frac{1}{p}\|x_m + \frac{1}{n}u\| + \frac{1}{np} < \frac{3}{np}\,. \qquad \ldots \text{(c)}
\end{aligned}$$

Since $x_m \in W_{n,p}$,

$$\begin{aligned}
&\left|\frac{F(x_m + (1/m)u) - F(x_m)}{1/m} - \frac{F(x_m + (1/n)u) - F(x_m)}{1/n}\right| \\
&\leq \left|\frac{F(x_m + (1/m)u) - F(x_m)}{1/m} - F'_+(x_m)(u)\right| \\
&+ \left|\frac{F(x_m + (1/n)u) - F(x_m)}{1/n} - F'_+(x_m)(u)\right| \\
&< \frac{2}{p}\,.
\end{aligned}$$

But

$$\begin{aligned}
&\left|\frac{F(x_m + (1/m)u) - F(x_m)}{1/m} - \frac{F(x_m + (1/n)u) - F(x_m)}{1/n}\right| \\
&\geq \left|\frac{F'_+(0)((1/m)(u+v)) - F'_+(0)((1/m)v)}{1/m} - \frac{F'_+(0)((1/n)u) - F'_+(0)((1/m)v)}{1/n}\right| \\
&- m|F'_+(0)((1/m)(u+v)) - F(x_m + (1/m)u)| - m|F'_+(0)((1/m)v) - F(x_m)| \\
&- n|F'_+(0)((1/n)u) - F(x_m + (1/n)u)| - n|F'_+(0)((1/m)v) - F(x_m)| \\
&\geq |F'_+(0)(u+v) - F'_+(0)(v) - F'_+(0)(u)| - \frac{n}{m}|F'_+(0)(v)| - \frac{4}{p} - \frac{3}{p} - \frac{3}{p} - \frac{3n}{mp} \\
&\hfill \text{from (a), (b) and (c)} \\
&\geq \frac{16}{p} - \frac{n}{m}|F'_+(0)(v)| - \frac{12}{p} \geq \frac{3}{p}
\end{aligned}$$

for any choice of $m \geq np|F'_+(0)(v)|$. This is our contradiction, so we conclude that W is first category. □

A real-valued function F on an open subset A of a normed linear space X is said to be *strictly differentiable* at $x \in A$ if

$$\lim_{\substack{z \to x \\ \lambda \to 0^+}} \frac{F(z + \lambda v) - F(z)}{\lambda} \text{ exists for all } v \in X$$

and this limit is $F'(x)(v)$, where $F'(x)$ is a continuous linear functional on X. Further, F is said to be *uniformly strictly differentiable* at $x \in A$ if this limit exists and is approached uniformly for all $v \in X$, $\|v\| \leq 1$.

For a real-valued function on an open subset of a normed linear space the set of points where the function is Fréchet differentiable but not uniformly strictly differentiable is first category in its domain, [Z1, p. 158]. So using this fact we can make the following deduction.

Corollary 2.1. *Given a real-valued function F on an open subset A of a normed linear space X, the set of points where F is Fréchet directionally differentiable but not uniformly strictly differentiable is first category in A.*

For a real-valued function on an open subset of a normed linear space the set of points where the function is both Fréchet subdifferentiable and Fréchet directionally differentiable contains the set of points where the function is Fréchet differentiable. So from Theorem 2 and Zajíček's result for Fréchet subdifferentiability, [Z2, p. 485], we can produce the following relation between points of Fréchet subdifferentiability and Fréchet directional differentiability.

Corollary 2.2. *Consider a real-valued function F on an open subset A of a normed linear space X.*

(i) *The set of points where F is Fréchet directionally differentiable but not Fréchet subdifferentiable is first category in A.*

(ii) *If F is lower semi-continuous on A and X is an Asplund space then the set of points where F is Fréchet subdifferentiable but not Fréchet directionally differentiable is first category in A.*

We pointed out before Theorem 2 that there exists on ℓ_1 an equivalent norm which is Gâteaux differentiable away from the origin but is nowhere Fréchet differentiable. Now this norm is everywhere Fréchet subdifferentiable but Fréchet directionally differentiable only at the origin, so there is no obvious improvement to be made to Corollary 2.2 (ii). In fact for any Banach space X which is not an Asplund space there exists a continuous convex function ϕ on an open convex subset A of X where the set of points of Fréchet differentiability is first category in A. So the set of points where ϕ is Fréchet subdifferentiable but not Fréchet differentiable is residual in A and from Theorem 2, the set of points where ϕ is Fréchet subdifferentiable but not Fréchet directionally differentiable is residual in A. We should point out that for Lipschitz functions on the real line it may be that the set of points of Fréchet subdifferentiability is first category; there is a Lipschitz function ψ where the set of points of differentiability is first category, [GS2,

p. 210], so it is not possible that both ψ and $-\psi$ are Fréchet subdifferentiable on a residual set.

For convex functions there is a useful continuity characterization of Fréchet directional differentiability which has had interesting geometrical consequences. Given a continuous convex function ϕ on an open convex subset A of a normed linear space X, the *subdifferential* of ϕ at x is the set

$$\partial\phi(x) = \{f \in X^* : f(y) \leq \phi'_+(x)(y) \ \text{ for all } \ y \in X\}.$$

We say that the subdifferential mapping $x \mapsto \partial\phi(x)$ is *restricted norm upper semi-continuous* at $x \in A$ if given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\partial\phi(y) \subset \partial\phi(x) + \epsilon B(X) \ \text{ for all } \ y \in B(x;\delta).$$

The function ϕ is Fréchet directionally differentiable at $x \in A$ if and only if the subdifferential mapping $x \mapsto \partial\phi(x)$ is restricted norm upper semi-continuous at x, [GM, Theorem 3.2]. It has been shown that a Banach space X is an Asplund space if it has an equivalent norm which is Fréchet directionally differentiable on the unit sphere, [CP, p. 453].

It is instructive to see that there is no comparable continuity characterization of Fréchet directional differentiability for locally Lipschitz functions. Given a locally Lipschitz function ψ on an open subset A of a normed linear space X, the *Clarke subdifferential* of ψ at $x \in A$ is the set

$$\partial\psi(x) = \{f \in X^* : f(y) \leq \limsup_{\substack{z \to x \\ \lambda \to 0^+}} \frac{\psi(z + \lambda y) - \psi(z)}{\lambda} \ \text{ for all } \ y \in X\}.$$

Now ψ is strictly differentiable at $x \in A$ if and only if $\partial\psi(x)$ is singleton and is uniformly strictly differentiable at $x \in A$ if and only if the subdifferential mapping $x \mapsto \partial\psi(x)$ is single-valued and norm upper semi-continuous at x, [GS1, p. 374]. There exists a locally Lipschitz function which is Fréchet differentiable and strictly differentiable at a point but not uniformly strictly differentiable at that point, [GS1, p. 373]; so the function is Fréchet directionally differentiable at the point but its subdifferential is not restricted norm upper semi-continuous there. But also there exists a Lipschitz function on the real line whose subdifferential mapping is everywhere restricted norm upper semi-continuous but nowhere single-valued, [GS2, p. 210], so by Corollary 2.1 the set of points where the function is not Fréchet directionally differentiable is residual.

We might well ask how our Theorem 2 generalizes for Gâteaux differentiability. For locally Lipschitz functions on certain Banach spaces we have sufficient information about generic sets of points of differentiability to achieve a generalization.

Given a real-valued function F on an open subset A of a normed linear space X we say that F is *intermediately differentiable* at $x \in A$ if there exists a continuous linear functional f on X such that

$$\liminf_{\lambda \to 0^+} \frac{F(x + \lambda v) - F(x)}{\lambda} \leq f(v) \leq \limsup_{\lambda \to 0^+} \frac{F(x + \lambda v) - F(x)}{\lambda} \ \text{ for all } \ v \in X.$$

Clearly if $F'_+(x)(v)$ exists at $x \in A$ for all $v \in X$ and F is intermediately differentiable at x then F is Gâteaux differentiable at x.

A GSG space is a Banach space which contains a dense continuous linear image of an Asplund space. Every closed linear subspace of a GSG space is a weak Asplund space. It has been shown that a locally Lipschitz function ψ on an open subset A of a closed linear subspace X of a GSG space is intermediately differentiable on a residual subset of A, [FP, p. 733]. So we can make an immediate deduction.

Theorem 3. *For a locally Lipschitz function ψ on an open subset A of a closed linear subspace X of a GSG space, the set*

$$\{x \in A : \psi'_+(x)(v) \ \textit{ exists for all } \ v \in X \ \textit{ and } \\ \psi \ \textit{ is not Gâteaux differentiable at } \ x\}$$

is first category in A.

For locally Lipschitz functions on separable Banach spaces rather more can be stated. For a locally Lipschitz function ψ on an open subset A of a separable Banach space X the set

$$\{x \in A : \psi \ \text{ is Gâteaux differentiable at } \ x \ \text{ and } \\ \psi \ \text{ is not strictly differentiable at } \ x\}$$

is first category in A, [GS2, p. 210]. So from Theorem 3 we deduce the extended result.

Theorem 4. *For a locally Lipschitz function ψ on an open subset A of a separable Banach space, the set*

$$\{x \in A : \psi'_+(x)(v) \ \textit{ exists for all } \ v \in X \ \textit{ and } \\ \psi \ \textit{ is not strictly differentiable at } \ x\}$$

is first category in A.

We should make the following remarks about these results. Theorem 3 does not hold generally for all Banach spaces; on ℓ_∞ the continuous semi-norm p defined for $x = (x_1, x_2, \dots, x_n, \dots)$ by $p(x) = \limsup_{n\to\infty} |x_n|$ has $p'_+(x)(v)$ existing at each $x \in \ell_\infty$ for all $v \in \ell_\infty$, but p is nowhere Gâteaux differentiable, [Ph, p. 13]. Even for a locally Lipschitz function on an open interval of the real line, Theorem 4 has no obvious improvement; there exists a locally Lipschitz function ψ everywhere differentiable on (a, b) and which is strictly differentiable only on a set of less than full measure, [M, p. 975].

References

[BP] Borwein J.M., Preiss D., *A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions*, Trans. Amer. Math. Soc. **303** (1987), 517–527.

[CP] Contreras M.D., Paya R., *On upper semi-continuity of duality mappings*, Proc. Amer. Math. Soc. **121** (1994), 451–459.

[FP] Fabian M., Preiss D., *On intermediate differentiability of Lipschitz functions on certain spaces*, Proc. Amer. Math. Soc. **113** (1991), 733–740.

[GM] Giles J.R., Moors W.B., *Generic continuity of restricted weak upper semi-continuous set-valued mappings*, Set-valued Analysis, to appear.

[GS1] Giles J.R., Scott Sciffer, *Continuity characterizations of differentiability of locally Lipschitz functions*, Bull. Austral. Math. Soc. **41** (1990), 371–380.

[GS2] Giles J.R., Scott Sciffer, *Locally Lipschitz functions are generically pseudo-regular on separable Banach spaces*, Bull. Austral. Math. Soc. **47** (1993), 203–210.

[M] Marcus S., *Sur les fonctions dérivées, intégrables, au sens de Riemann et sur les dérivées partialles mixtes*, Proc. Amer. Math. Soc. **9** (1958), 973–978.

[Ph] Phelps R.R., *Convex Functions, Monotone Operators and Differentiability*, Springer-Verlag, Lecture Notes in Math. **1364**, 2nd ed., 1993.

[Z1] Zajíček L., *Strict differentiability via differentiability*, Act. U. Carol. **28** (1987), 157–159.

[Z2] Zajíček L., *Fréchet differentiability, strict differentiability and subdifferentiability*, Czech. Math. J. **41** (1991), 471–489.

The University of Newcastle, NSW 2308, Australia

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